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EFFECTS OF FRINGING FIELDS ON SINGLE PARTICLE DYNAMICS

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ABSTRACT

General formulation of magnetic field expressions valid for multipoles of any order are reported. Analytical expressions derived from the cylindrical model are shown. The application on DAΦNE optics is described.

1. Multipolar Magnetic Field General Expressions

The scalar magnetic potential generated by a quadrupolar field not depending on *z* is:

$$P(x,y) = G\,xy$$

from which by applying the gradient operation the magnetic field components are derived:

$$B_x = G\,y$$

$$B_y = G\,x$$

For a quadrupolar field depending on *z* the condition of satisfying Maxwell equations in three dimensions requires the dependence of the function *G* not only on *z*, but also on *r*:

$$P(r,\theta,z) = G\,xy = G(z,r)\,r\cos\theta\,r\sin\theta = G(z,r)\,r^2\,\frac{\sin 2\theta}{2!}$$

where $G(r,z)$ must be of the form:

$$G(z,r) = G_{20}(z) + G_{22}(z)r^2 + \dots$$

This formalism can be extended to any multipole of order m:

$$P_m(r,\theta,z) = \frac{r^m \sin m\theta}{m!}\left[G_{m0}(z) + G_{m2}(z)r^2 + \dots\right]$$

Substituting $\sin m\theta$ with $\cos m\theta$ the multipole is skew. In particular with $\cos m\theta,\, m = 0$ the solenoid is represented:

$$P_S(r,z) = G_{S0}(z) + G_{S2}(z)r^2 + \dots$$

The function $G_{m0}(z)$ contains all the information on the magnetic field of the multipole, since the $G_{mp}(z)$ are from it derived:

$$G_{m\,2p}(z) = (-1)^p \frac{m!}{4^p (m+p)!\,p!} \frac{d^{2p} G_{m0}(z)}{dz^{2p}}$$
$$G_{m\,2p+1}(z) = \frac{dG_{m\,2p}(z)}{dz}$$

and therefore the magnetic field of any multipole of order m written in cylindrical coordinates is:

$$B_r(r,\theta,z) = \frac{\sin m\theta}{m!} \sum_{p=0}^{\infty} (m+2p) G_{m\,2p}(z) r^{2p+m-1}$$
$$B_\theta(r,\theta,z) = \frac{\cos m\theta}{(m-1)!} \sum_{p=0}^{\infty} G_{m\,2p}(z) r^{2p+m-1}$$
$$B_z(r,\theta,z) = \frac{\sin m\theta}{m!} \sum_{p=0}^{\infty} G_{m\,2p+1}(z) r^{2p+m}$$

For example the cylindrical components of a horizontal dipole ($m = 1$) magnetic field are:

$$B_r = \sin\theta \left\{G_{10}(z) + 3G_{12}(z)\, r^2 + \dots\right\}$$
$$B_\theta = \cos\theta \left\{G_{10}(z) + G_{12}(z)\, r^2 + \dots\right\}$$

from where the Cartesian components are:

$$B_x = B_r \cos\theta - B_\theta \sin\theta = 2G_{12}(x^2 + y^2) + ...$$
$$B_y = B_r \sin\theta + B_\theta \cos\theta = G_{10} + G_{12}(x^2 + 3y^2) + ...$$

and the characteristic function is the vertical magnetic field on the axis:

$$G_{10}(z) = B_y(0,0,z)$$

For a quadrupole (m = 2) it is the gradient on the axis:

$$G_{20}(z) = \left.\frac{\partial B_y}{\partial x}\right|_{r=0}$$

while for a solenoid the longitudinal magnetic field is the first derivative with respect to z of the magnetic scalar potential

$$G_{S1}(z) = B_z(z) = \frac{dG_{S0}}{dz}$$

The characteristic function of a real magnet is in principle measurable.

2. Analytical Model (Cylindrical Model)

An analytical expression fitting the characteristic function of a magnet is a powerful tool for any kind of analysis of the magnet characteristics and of its effects on the optics. This is what is obtained with the so called *cylindrical model*[1]: the field near the axis of a multipole created by a cylindrical sheet of currents has an analytical expression defined by three parameters: the length of the cylinder, $2Z_L$, its radius R and the circulating peak current I_c. The analytical description of $G_{m0}(z)$ and its derivatives is possible. In the following table the characteristic functions $G_{S0}(z)$ for the solenoid, $G_{10}(z)$ for the dipole and $G_{20}(z)$ for the quadrupole are shown, using the definition of the functions:

$$f_h(t) = \left(\frac{t}{\sqrt{R^2 + t^2}}\right)^h$$

Table I - Characteristic functions of some multipoles

Multipole	m	G_{m0}
Solenoid	0	$B_z(z) = G_{S1}(z) = \frac{\mu_o I_c}{4Z_L} \; f_1(t)\Big\vert_{z-Z_L}^{z+Z_L}$
Dipole	1	$G_{10}(z) = \frac{\mu_o I_c}{4R}\left[2 f_1(t) - f_3(t)\right]_{z-Z_L}^{z+Z_L}$
Quadrupole	2	$G_{20}(z) = \frac{\mu_o I_c}{8R^2}\left[9 f_1(t) - 8 f_3(t) + 3 f_5(t)\right]_{z-Z_L}^{z+Z_L}$

A quality of the functions f_{2k+1} is that their first and second derivatives can be expressed through the same functions by the relationships:

$$g_{2k+1}(t) = \frac{df_{2k+1}(t)}{dt} = \frac{(2k+1)R^2}{\left(R^2+t^2\right)^{3/2}} f_{2k}(t)$$

and

$$\frac{d^2 f_{2k+1}}{dt^2} = (4k^2+2k) f_{2k-1} - (12k^2+12k+3) f_{2k+1} + \\ + (12k^2+18k+6) f_{2k+3} - (4k^2+8k+3) f_{2k+5}$$

so all the higher order terms are linear combinations of f_{2k+1} and g_{2k+1}.

Once measured the magnet characteristic function, it is possible to fit it with one or a superposition of CM, using as fitting parameters the three CM ones: length, radius and current. From there on all field terms are analytically defined. The example of the KLOE[2] detector solenoid is shown in the following figures and table. Three CM with different fitting parameters (see Table II) describe the field on axis with good approximation. In Fig. 1 the dots represent the measurement and the solid line the analytical fit. The goodness of the fit is shown in Fig. 2 where the difference between the longitudinal component of the magnetic field along *z* at two different values of *r* (*r* = 0., and *r* = 7.5 cm) is shown: this difference follows the second derivative of the field on axis:

$$B_z(r,z) = G_{S1}(z) + G_{S3}(z) r^2 + ...$$

Table II: *CM fitting parameters for KLOE detector solenoid*

	Z_L(m)	R(m)	I(A)
1	1.9707	0.1594	$2.68\ 10^7$
2	2.1175	0.1705	$-0.26\ 10^7$
3	0.7480	1.3163	$-0.30\ 10^7$

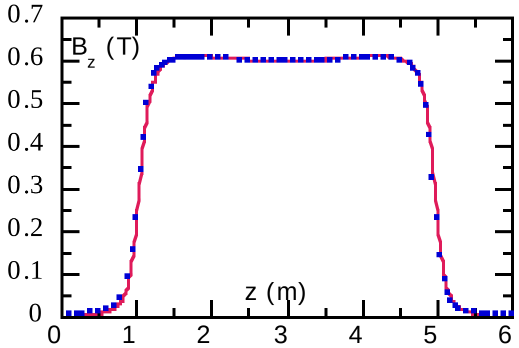

Figure 1: *Longitudinal magnetic field in KLOE detector solenoid*

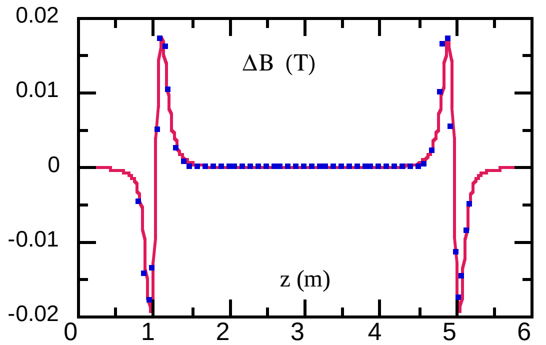

Figure 2: $B_z(r,z) - B_z(0,z)$, $r = 0.075\ m$ *in KLOE detector solenoid*

3. Effects on Linear Optics

Linear magnetic elements at the lowest approximation are usually described by the rectangular model. In reality the characteristic function $G_{m0}(z)$ is far from a step-wise function (see Fig. 3). A more realistic description is obtained by 'slicing' the characteristic function, either measured or analytically described by the CM, into a reasonable number of intervals[3], each of them described by a rectangular model matrix (see Fig. 4). The product of all these matrices represents the total matrix.

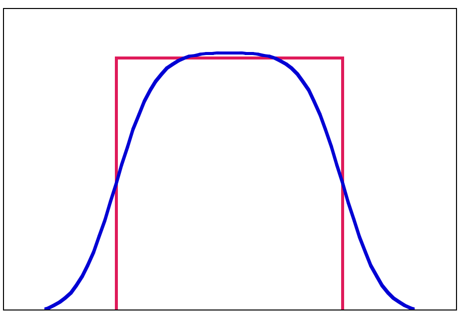

Figure 3: *Rectangular model and measured* G_{m0}

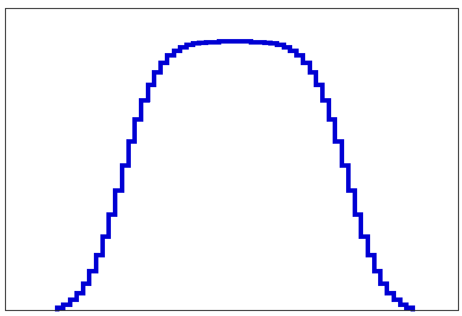

Figure 4: *Sliced computation of the measured* G_{m0}

In the case of a quadrupole, the final matrix can be written again as a *quadrupole like* matrix, in which the two characteristic parameters (length and gradient) are modified differently for the two planes; the quadrupole needs therefore four parameters. The differences between the rectangular model and the *real* one are of course related to the G_{20} behaviour, which is strongly correlated to the ratio R/Z_L. These differences are consequence of the linear fringe field and are totally different from the non linear fringing field effects coming from the upper terms $\left(G_{mn}(z),\, n>0\right)$.
The linear fringing field lowers the gradient in the focused plane and make it stronger in the defocused one. It is straightforward, by applying:

$$\Delta Q = \frac{1}{4\pi}\beta\,\Delta KL$$

to show that the total effect is a negative tune shift. In fact in the plane where the quadrupole focuses the value of |K| decreases and the corresponding ΔQ is negative; in the plane where the quadrupole defocuses the value of |K| increases and again ΔQ is negative. The effect is of course stronger in the plane with higher chromaticity. For example in the DAΦNE main rings arcs, which contain two different kinds of quadrupoles, the value of ΔQ is about 0.01 (H), 0.02 (V).
For a solenoid, whose matrix can be written as the product of a rotation matrix and a focusing one[4] the method can be equally applied to the focusing matrix, while the rotation matrix remains unchanged.

4. DAΦNE Interaction Regions

In the DAΦNE Interaction Regions the magnetic field is a superposition of quadrupole and solenoid fields (see Fig. 5).

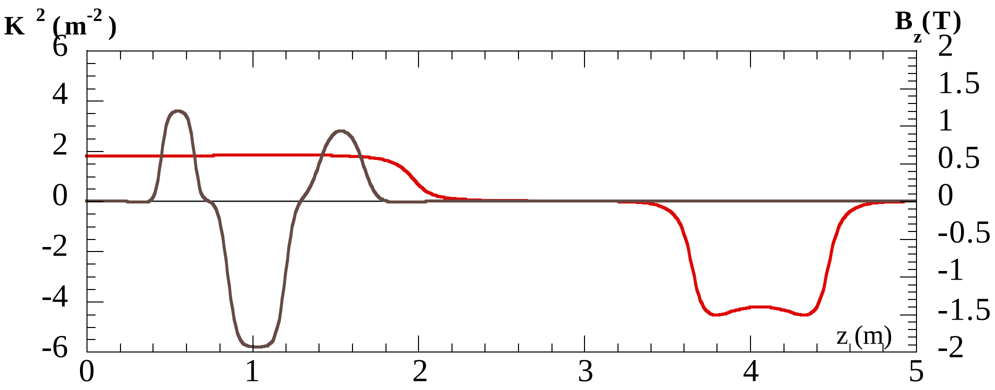

Figure 5: *KLOE half IR. Behaviour of low beta quadrupole gradients and longitudinal magnetic fields of detector and compensator.*

The solenoidal compensation is done applying the Rotating Frame Method[4], in which the quadrupoles are tilted following the rotation of the transverse plane introduced by the solenoids.
The field components of a quadrupole tilted by an angle θ_T are:

$$B_r(r,\theta,z) = \sin 2(\theta+\theta_T)\left\{G_{20}(z)\, r + G_{22}(z)\, r^3 + \ldots\right\}$$
$$B_\theta(r,\theta,z) = \cos 2(\theta+\theta_T)\left\{G_{20}(z)\, r + G_{22}(z)\, r^3 + \ldots\right\}$$
$$B_z(r,\theta,z) = \frac{\sin 2(\theta+\theta_T)}{2}\left\{G_{21}(z)\, r^2 + G_{23}(z)\, r^4 + \ldots\right\}$$

Beams travel off-axis in the IRs since they cross with a tunable horizontal angle θ_{cross}. Particle trajectories are computed by integrating the equations of motion along the IR. The linear jacobian around the trajectory on axis is in agreement with the linear IR transport matrix computed with the *sliced* computation. Information about phase advance and optical functions at the IR ends are deduced from the jacobian for the different crossing angles. For the nominal optical parameters at the IP (β_x = 4.5 m, β_y = 4.5 cm), as θ_{cross} increases the phase advance along the IR increases, especially in the vertical plane, this because around the off axis trajectory quadrupoles add an alternate bending action, like a wiggler, giving vertical focusing (see fig.6). In the presence of solenoids this focusing acts in the plane perpendicular to the trajectory plane point by point. At the IR end, where the normal modes become horizontal and vertical because of the RFM method, the increase in phase advance appears in the vertical plane.

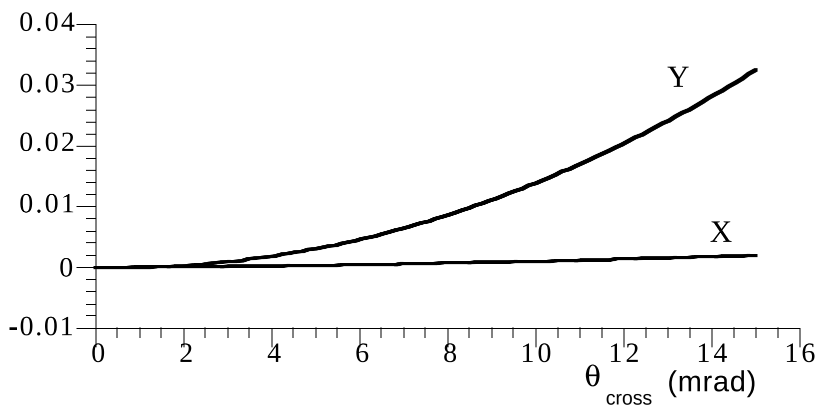

Figure 6: *Tune shift with crossing angle in DAΦNE optics*

The R.F.M. applied on axis is exactly valid off-axis only in the linear approximation. The four parameters used for decoupling the motion between the IP and the IR end[4], i.e. the three quadrupole tilting angles and the integral of the longitudinal magnetic field on the compensator solenoid, depend in principle on the crossing angle. Using the linear terms of the jacobian computed around the trajectory, it is possible to readjust the values of the decoupling parameters by minimizing the non diagonal elements of the jacobian. The differences between the decoupling parameters with θ_{cross} in the nominal range have resulted smaller than the alignment tolerances.

References

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4. M. Bassetti, M.E. Biagini, C. Biscari, *Solenoidal Field Compensation*, these proceedings.