Frascati Physics Series Vol. X (1998), pp. 289-296
14th Advanced ICFA Beam Dynamics Workshop, Frascati, Oct. 20-25, 1997

A BEAM-BEAM TUNE SHIFT SEMI-EMPIRICAL FIT

M. Bassetti, M.E. Biagini
INFN-LNF, C.P. 13, 00044 Frascati (RM),Italy

ABSTRACT

A new approach to the beam-beam limit, taking into account the effect of the energy fluctuations and of the electric field work on colliding particles, is presented. A formula is derived by a fit of data collected at several colliders with energies ranging in two orders of magnitude.

1. Introduction

The comprehension of the so-called "beam-beam limit" is one of the major problems in the theory of electron storage rings. By squeezing the beams we can theoretically get an infinite luminosity, but this actually never happens. For given values of the machine parameters we find instead a limit. Once working energy, beam current and machine optics are fixed, to optimise the luminosity there is only one degree of freedom left, that we can choose among many related parameters as the vertical tune shift ξ_y, the ratio "r" between the rms beam sizes σ_y and σ_x, the coupling parameter "κ", etc. A first study to the beam-beam limit, the so-called "Amman-Ritson" optical interpretation[1)], states that the limit should be the same on ξ_x and ξ_y and independent on the energy, if the working point is chosen far away from resonance lines. The first experimental results on the ADONE storage ring, in Frascati, showed[2)] instead a violent dependence on the energy below 1 GeV. Spear, VEPP-2M and other colliders confirmed this behaviour[3)]. At the lowest energies colliders show a luminosity scaling $L \approx E^{6 \div 7}$ instead of the expected behaviour $L \approx E^4$. This seems to happen up to an energy characteristic of each collider, that we call threshold energy (TE). Above TE the tune shift limit slightly depends on the beam energy.

A new attempt for the comprehension of the beam-beam limit is presented in this paper. We limit ourselves to flat beams colliders, working above TE. We suppose that two phenomena are candidates to play an important role in the beam beam interaction:

- the energy fluctuation of the single particle which characterises the synchrotron radiation;
- the work done by the electric field component of a beam on each particle of the opposite beam.

2. Beam-beam tune shift from luminosity measurements

Usually published data in literature concern luminosity, emittance and beam current measurements only. To know other important machine parameters, needed for our analysis (as σ_x, σ_y, κ, ξ_x and ξ_y) we use the luminosity formula:

$$L = \frac{f_o n_b N^2}{4\pi\sigma_x\sigma_y} \tag{1}$$

Knowing L, the number of bunches n_b and the number N of particles per bunch, we can compute the beam cross section $\sigma_x\,\sigma_y$, even if the transverse beam sizes are not known separately. In the most general case of non zero dispersion at the I.P., it is:

$$\sigma_x = \sqrt{\left[\varepsilon_s + \varepsilon\,(1-q)\beta_x\right]} \qquad\qquad \sigma_y = \sqrt{\varepsilon\beta_y q} \tag{2}$$

where ε is the off coupling betatron emittance, ε_s the synchrotron contribution to the emittance of the dispersion at the I.P. and $q = \kappa/(1+\kappa)$. With a little bit of algebra we can deduce the coupling factor κ and the beam sizes from L. Then the tune shifts can be computed by the well known formula:

$$\xi_{x,y} = \frac{r_e N\beta_{x,y}}{2\pi\gamma\,\sigma_{x,y}\sigma_x(1+\sigma_y/\sigma_x)} \tag{3}$$

For our fit we will use the tune shifts as computed from eq. (3), rather than the measured ones: in fact it is worthwhile to point out that often the experimental vertical tune shift is computed from the ratio L/I, neglecting the beam sizes contribution from σ_y/σ_x in eq. (3).

3. A guiding hypothesis

Let us consider now some luminosity data available in literature[4,5], some of them being unfortunately very old. We consider for our purposes the aformentioned variables:

1) the average energy fluctuations ΔE_F of each particle between two successive interactions. From Ref. 6) we have:

$$\Delta E_F\ [\mathrm{KeV}] = 16.11294\ \frac{E\ [\mathrm{GeV}]^{3.5}}{\rho_F[\mathrm{m}]\ \sqrt{N_i}} \qquad (4)$$

   Ni is the number of Interaction Points (I.P.) per turn and ρ_F is the equivalent machine bending radius in presence of dipoles and wigglers[7] ($\rho_F = \sqrt{2\pi/I_3}$);

2) the work W_{el} done by the electric field of a beam on particles of the opposite beam. This work exists also when the beams collide head-on, because every particle trajectory has a slope at the I.P. which creates an electric field component parallel to the trajectory itself. W_{el} acts on both planes. It is approximately:

$$W_{el} \approx \frac{1}{2} m_o\ c^2\ \gamma\ [\ <\Delta x'^2>\ \sigma'^{\,2}_x + <\Delta y'^2>\ \sigma'^{\,2}_y]^{1/2} \qquad (5)$$

   In fact $<\Delta x'^2>^{1/2}$ and $<\Delta y'^2>^{1/2}$ are proportional to the transverse electric field of the beam-beam and σ'_x and σ'_y determine the fraction of the electric field component that acts on the particle.

Our guiding hypothesis is that W_{el} can create a mechanism of instability and the noise ΔE_F can neutralise this mechanism. We may compare W_{el} to a radio signal with very high correlation and ΔE_F to a noise. ΔE_F must be large enough to cancel out the signal W_{el}. We expect ΔE_F to be much larger than W_{el}, due to the very sharp Fourier spectrum of W_{el} with respect to the very wide ΔE_F one.
For the $<\Delta x'^2>$ and $<\Delta y'^2>$ values we assume the results deduced in Ref. 8),9):

$$<\Delta x'^2> \ = \ <\Delta y'^2> \ = \ \frac{4 r_e^2 N^2}{\gamma^2 \sigma_x^2}\ \eta_o(r) \qquad (6)$$

where $r = \sigma_y/\sigma_x$ and the function $\eta_0(r)$ [8] is:

$$\eta_0(r) = \frac{1}{\sqrt{D}}\ \arctan\left[\frac{\sqrt{D}}{Q}\right] \qquad (7)$$

with D and Q defined by:

$$D = 3\, r^4 - 10\, r^2 + 3 \qquad (8)$$

$$Q = 3\, r^2 + 8\, r + 3 \qquad (9)$$

The function $\eta_0(r)$ has an analytical continuation for D<0: $\sqrt{D}$ becomes $\sqrt{abs(D)}$ and the function *arctan* becomes *arctanh*. For each value of r between 0 and 1 a good approximation is:

$$\eta_0(r) = \frac{0.302}{1 + 2.2\, r + r^2} \qquad (10)$$

If there is a synchrotron emittance at the IP, $\eta_0(r)$ becomes:

$$\eta(r, \frac{\beta_x}{\beta_y}, \frac{\varepsilon_s}{\varepsilon}) = \eta_0(r)\ \frac{(1 + r^2\, \beta_x / \beta_y)}{(1 + \varepsilon_s / \varepsilon)} \qquad (11)$$

We can finally write the electric field work as:

$$W_{el} \approx \frac{m_o c^2 r_e N}{\sigma_x} \sqrt{\eta(r)}\ \sigma' \qquad (12)$$

where σ' is the beam slope at the interaction:

$$\sigma' = \sqrt{\sigma'^2_x + \sigma'^2_y} \qquad (13)$$

4. A semi-empirical fit

We come finally to our prediction about the vertical tune shift, through a phenomenological fit of the vertical tune shifts as deduced by the published data.
For each collider and for each different fit we consider the parameter λ_i defined as:

$$\lambda_i = (\xi_y^{meas})_i / (\xi_y^{fit})_i \qquad (14)$$

The most elementary formula we can think of is a fit with a constant, from which we get the ξ_y value:

$$\xi_y = \xi_y^{aver}\, (1 \pm \sigma_o) \qquad (15)$$

where $\xi_y^{aver} = 0.037$ and $\sigma_o = \langle\lambda_i^2\rangle - \langle\lambda_i\rangle^2 = 0.274$.

Then we can consider a simple formula as:

$$\xi_y = \text{cost} \cdot (A_1^{e1}) \cdot (A_2^{e2}) \ldots\ldots \quad (16)$$

where $(A_1, A_2, A_3, \ldots)$ are parameters that we choose among the optical parameters of each collider. The exponents e_i can be found by applying the least square method to the function:

$$G = \Sigma_i [\ln \lambda_i]^2 \quad (17)$$

By applying our guiding hypothesis, we should fit ξ_y as:

$$\xi_y = \text{cost} \frac{\Delta E_F^{e1}}{W_{el}^{e2}} \quad (18)$$

However there is a disavantage in this formula, since both W_{el} and ξ_y depend on the number of particles N. Hence we would have an implicit equation in N or ξ_y not suitable for fitting.
We consider then ξ_y as a function of parameters as E, $\eta(r)$, σ', ρ_F and N_i that appear in the ΔE_F and W_{el} formulae. Computing the exponent of each parameter separately we get:

$$\xi_y = 5.80 \times 10^{-3} \frac{E^{.26} \eta(r)^{.54}}{(\sigma')^{.40} \rho_F^{.23} N_i^{.12}} \quad (19)$$

Since the exponent of N_i is about one half the ρ_F one, namely ρ_F and N_i act as $(\rho_F \sqrt{N_i})$, in agreement with our hypothesis on the role of the fluctuations, we can use $(\rho_F \sqrt{N_i})$ as an independent variable and we get:

$$\xi_y = 5.87 \times 10^{-3} \frac{E^{.27} \eta(r)^{.55}}{(\sigma')^{.40} (\rho_F \sqrt{N_i})^{.24}} \quad (20)$$

Eq. (20) also shows the dependence of ξ_y on σ', that we interpret as a negative effect on ξ_y of the electric field. As a comparison, if we consider as an independent variable the contribution from the damping, namely $(\rho_D N_i)$, where $\rho_D = \sqrt{(2\pi/I_2)}$, we get instead:

$$\xi_y = 5.15 \times 10^{-3} \frac{E^{.20} \eta(r)^{.48}}{(\sigma')^{.40} (\rho_D N_i)^{.18}} \quad (21)$$

Let's now compare the results. The goodness of the fit is given by the Snedecor factor: $F = (\sigma_o/\sigma^{fit})^2$. Smaller is σ^{fit} with respect to σ_o, larger is F and smaller is the probability P that the choice of the fit parameters is unreliable from a physical point of view. For a comparison, Table 1 reports the values obtained for all the formulae used.

Table 1: *Fits comparison*

FORMULA	σ^{fit}	**F**	**P**
(15)	.274	1.	.5
(21)	.112	5.6	3.4×10^{-4}
(19)	.108	6.4	2.1×10^{-4}
(20)	.108	6.4	1.5×10^{-4}

Formula (20) gives the best fit results.Their values for the considered colliders together with the measured tune shifts and the ratios λ_i are listed in Table 2.

Table 2: *Fit results from formula (20)*

RING	ξ_y^{MEAS}	ξ_y^{FIT}	λ_i
VEPP-2M	0.0369	0.0397	0.930
VEPP-2MW	0.0367	0.0381	0.962
ADONE	0.0463	0.0497	0.932
BEPC-1	0.0276	0.0279	0.988
BEPC-2	0.0361	0.0301	1.200
BEPC-3	0.0399	0.0317	1.257
SPEAR2	0.0280	0.0317	0.883
VEPP-4	0.0578	0.0551	1.048
AR-TRISTAN	0.0651	0.0646	1.007
DORIS2	0.0289	0.0282	1.022
CESR-1	0.0240	0.0224	1.068
CESR-2	0.0258	0.0299	.862
CESRA	0.0386	0.0345	1.112
PETRA-1	0.0223	0.0265	0.840
PETRA-2	0.0344	0.0350	0.984
PEP-1	0.0443	0.0395	1.122
PEP-2	0.0412	0.0431	0.957
TRISTAN	0.0325	0.0401	0.810
LEP-45.6	0.0439	0.0416	1.055
LEP-86	0.0394	0.0409	0.962
LEP-86B	0.0488	0.0464	1.051
LEP-91.5	0.0488	0.0458	1.065

In Fig.1 for each collider the fit results (black triangles) are compared with the measured ξ_y values (white squares).

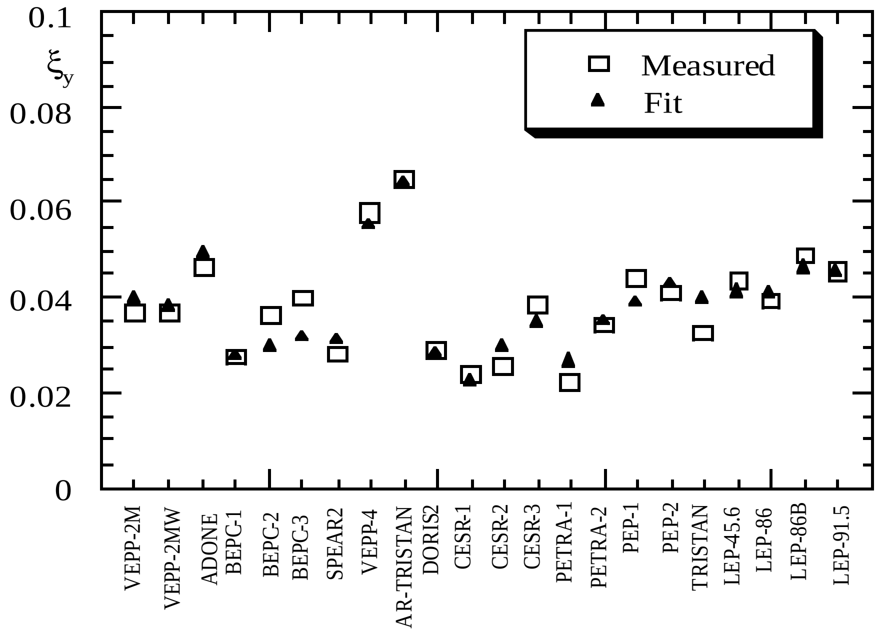

Figure 1: *Comparison of fit and measured tune shifts for flat beams colliders. Machines are listed in order of increasing energy.*

If we apply our formula (20) to the DAΦNE Φ-factory now under commissioning in Frascati we have: $(\xi_y^*)_{design} = 0.04$, $(\xi_y^*)_{fit} = 0.049$.

5. Discussion

Let's make some remarks on the results of Table 1. The fit with eq. (20) reduces the dispersion on the tune-shift data to 11% from 27% of the fit with a constant. Moreover we would like to point out that:

1) the considered parameters range on a large scale: for example between VEPP-2M and LEP the machine radius ranges from 1m to 2700 m, the energy from .5 GeV to 91 GeV, the number of particles/bunch from $7.x10^9$ to $3x10^{11}$.

2) There is a strong dependence on the betatron slope. Without the hypothesis on the work of the electric field this dependence would be unpredictable.

3) The difference in probability and dispersion using the factor ($\rho_F \sqrt{N_i}$) instead of ($\rho_D N_i$) stresses on the role of the fluctuations instead of the damping.

4) The tunes do not appear in eq. (20), since they did not contribute to improve the fit. Our opinion is that the maximum luminosity is reached by experimentally moving the tunes in order to stay far from resonances. At that point the beam-beam limit does not depend anymore on the working point but on other phenomena.

5) Not always maximum tune shift corresponds to maximum luminosity: we want to fit the maximum tune shift, while the published data are often those relative to the luminosity record.

Acknoledgements

We wish to thank Dr. H. Burkhardt from CERN for kindly providing the LEP data.

References

1. F.Amman, D.Ritson, Frascati Internal Report LNF- 61-38 (1961)

2. S. Tazzari, AIP Conf. Proc. **57**, (1979)

3. H. Wiedemann, AIP Conf. Proc. **57**, (1979)

4. A.Chao, SLAC-PUB-3825, (1985)

5. J.T.Seeman, SLAC-PUB-3825, (1985)

6. M.Sands, SLAC-121, (1970)

7. R.M.Helm *et al*, IEEE Trans. Nucl. Sci. **NS-20**, 900 (1973)

8. M.Bassetti, M.Gygi-Hanney, CERN/LEP-221, (1980)

9. M.Bassetti, M.Gygi-Hanney, CERN/LEP-447, (1983)