DAΦNE TECHNICAL NOTE

INFN - LNF, Accelerator Division

Frascati, March 13, 1992

Note: **LC-4**

SOME REMARKS ON DAΦNE POSITRON LINAC FOCUSING

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1- INTRODUCTION

Usually the focusing system for positron linacs is composed of:

1. Matching device transforming the point like positron source with large angular divergences into small angle divergences and larger lateral dimensions acceptance of the accelerating structure. Two types of matching devices are commonly used:
   - Quarter wave transformer (QWT)
   - Flux concentrator with adiabatically changing magnetic field.

2. Focusing along the linear accelerator.

Theoretically, different focusing devices can be used for this purpose: solenoids, quadrupole singlets, doublets and triplets. In practice, however, the most frequently used are solenoids and FODO quadrupoles. The efficiency of a quadrupole focusing is higher than the one of a solenoid. The solenoidal magnetic field should exist uniformly all along the linear accelerator, whereas the previous one is concentrated only in quadrupoles and the distance between the quadrupole lenses increases monotonically with energy rise. However, as it will be seen below, the quadrupole focusing cannot be applied for low energies of the beam, so the solenoidal focusing is indispensable.

In practice (SLAC, DESY, CERN) the quadrupole focusing is used for energies of the order of 100 MeV and higher on. This limitation is mainly connected with the constructional requirements for quadrupoles and finite length of accelerating sections. For low energies, the quadrupoles should be placed very close one to each other, since the length of the accelerating structure is usually of the order of few meters, it is necessary that the quadrupole aperture includes the accelerating structure, so that the diameter of a quadrupole should be about 15 cm for a 3 GHz structure. To get the proper quadrupole field, the quadrupole length must be about twice its diameter so that the length should be greater than (25-30) cm. Taking into account the mechanical constraints, the distance between quadrupoles cannot be smaller than (25-30) cm. This altogether imposes a limit on the low energy end of the FODO system, of about 100 MeV. For lower energies, rather solenoidal focusing should be used, since at low energy the phase acceptance of a solenoid can be higher.

2. MATCHING DEVICES

2.1 The Quarter Wave Transformer

The system is composed of a short lens with a high magnetic field of the order of (1.5÷2) T placed directly beyond the e^- e^+ converter followed by a long solenoidal magnetic field imposed over the first few accelerating sections. Usually the short lens is pulsed. (DESY, FRASCATI, LEP, KEK). The name of the transformer comes from the fact that the Larmor angle of a particle inside the lens is $\pi/2$ ie. corresponds to 1/4 of Larmor wave-length.

2.1.1 Radial Acceptance

Radial acceptance of the system is given by [1]

$$R_o = R_b\, B_s / B_o \tag{1}$$

where

R_o - is the radius of the beam at the converter
R_b - is the radius of the beam in the accelerating section. Usually it is assumed that $R_b \approx (0.8 \div 0.9)$ of the smallest iris aperture
B_o - is the magnetic field in the short lens
B_s - is the magnetic field in the solenoid

Assuming - B_o = 2 T, B_s = 0.4 T, R_b = 0.9 cm, we obtain R_o = 0.18 cm.

For the design average positron energy out of the converter E_{av} = 8 MeV the magnetic length of the lens, corresponding to the Larmor angle equal to $\pi/2$, is:

$$L_m = 1.7045\ 10^{-3}\, \pi\, \gamma / B_o = 4.5 \text{ cm}. \tag{2}$$

Where $\gamma = m/m_o$ is the relativistic factor.

2.1.2 Angular Acceptance

The maximum accepted angle at the converter is given by [1]

$$\phi_{max} = \frac{e\, B_o\, R_b}{2P}\,(1 + B_s/B_o) \tag{3}$$

For momentum P = 8 MeV/c and above given values for R_b, Bo, Bs, ϕ_{max} = 21.8°.

2.1.3 Energy Acceptance

For a point source the total width at midheight of the energy band is approximately [2]

$$\Delta E = \frac{4\ E_c\ B_s}{\pi\ B_o} \quad = 2\ \text{MeV} \tag{4}$$

This value is rather too low for our converter and we can have some difficulties to get enough positrons. According to our requirements, at the end of the linac we should have the 510 MeV positrons with the total energy spread of ± 1% i.e. $\Delta E_t = \pm\ 5$ MeV. The total energy spread can be expressed as:

$$\Delta E_t = \sqrt{\Delta E_s{}^2 + \Delta E\varphi^2 + \Delta E_{bl}{}^2} \tag{5}$$

where - ΔE_s represents energy dispersion of the source (converter)
- ΔE_φ is due to phase difference in linac
- ΔE_{bl} corresponds to beam loading in linac.

The energy dispersion ΔE_{bl} due to the beam loading for positrons is negligible. The two other components can be equally important for the positrons production, since the electron intensity on the converter is proportional to ΔE_φ and the number of positrons is proportional to ΔE_s. For a first estimation we can put $\Delta E_s = \Delta E_\varphi$ and we obtain:

$$\Delta E_s = \Delta E_\varphi = \Delta E_t/\sqrt{2} \ = \pm\ 3.5\ \text{MeV}.$$

The total energy band accepted from the converter in the case of the central energy E_c = 8 MeV can be (4.5 - 11.5) MeV.

We should also check what will be the positron phase acceptance corresponding to the above energy dispersion. First we will show that since the positrons velocities for energies E ≥ 5 MeV are close to that of light there will be only small phase changes of positrons during acceleration in the linac so that their phase dispersion will correspond closely to that of driving electrons (we do not consider here a special case when the positrons are first decelerated and then accelerated so that some phase mixing can occur).

The longitudinal equations of motion for TW accelerating structure can be written in the form:

$$d\gamma/ds = A\cos\varphi \tag{6}$$

$$d\varphi/ds \ = 2\,\pi\,(1/\beta_s - 1/\beta) \tag{7}$$

where $\gamma = m/m_o$
$A = e E \lambda / m_o c^2$
E = electric field intensity in the accelerating section
λ = wavelength
$\beta = v_z/c$ - particle velocity
β_s = phase velocity in the structure. Usually $\beta_s = 1$, except in the buncher
$s = z/\lambda$

The first integral of these equations can be written in the form:

$$\frac{\gamma}{\beta_s} - \sqrt{\gamma^2-1} = \frac{A}{2\pi}\sin\varphi + C1 \tag{8}$$

Denoting by subscript "o" the initial values and putting $\beta_s = 1$ (the structure is uniform with $v_\varphi = c$) we obtain for the constant C1:

$$C_1 = \gamma_o - \sqrt{\gamma_o^2 - 1} - \frac{A}{2\pi}\sin\varphi_o$$

For changes of the phase φ we have then

$$\sin\varphi = \sin\varphi_o + \Delta \tag{9}$$

where

$$\Delta = \frac{2\pi}{A}\left[\gamma - \gamma_o - \left(\sqrt{\gamma^2 - 1} - \sqrt{\gamma_o^2 - 1}\right)\right] \tag{10}$$

In our case we have : $E \approx 20$ MeV/m, $\lambda \approx 0.1$ m, $\gamma_o \approx 10$, $\gamma \approx 1000$ and $\Delta = -7.97\ 10^{-2}$ corresponding to $\Delta\varphi \approx -4.6°$. It follows then that the phases of positrons in the linac remain practically constant and the phase difference at the end of the linac corresponds to the phase difference of electrons striking the converter.

Equation for $\sin\varphi$ gives also the asymptotic phase in the high energy linac. Since according to Eq. (7) the optimum phase for acceleration is 0° then the initial phase φ_c for the center of the bunch should be $\varphi_c \approx 0 + (-\Delta\varphi)$ to arrive at the end of the linac with the optimum phase.

We can calculate now the acceptable phase dispersion of positrons corresponding to the assumed energy dispersion $\Delta E_\varphi \approx \pm 3.5$ MeV.

Denote by: Average energy $W_{av} = (W_M + W_m)/2$

Average energy dispersion $\Delta W_{av} = (W_M - W_m)/2$

Relative energy dispersion $\Delta W_{av}/W_{av} = \dfrac{W_M - W_m}{W_M + W_m}$

Where W_M and W_m are maximum and minimum energy correspondingly.

Generally we have:

$$W = W_o \cos\varphi$$

Assuming that the optimum phase is zero and the maximum acceptable phase is φ_M we have:

$$\frac{\Delta W\varphi}{W_{av}} = \frac{3.5}{500} = \frac{1 - \cos\varphi_M}{1 + \cos\varphi_M} = \delta$$

and

$$\varphi_M = \arccos\left(\frac{1 - \delta}{1 + \delta}\right) = 9.57° \qquad (11)$$

Thus, it has been verified that taking the electron phase dispersion of ±8° we can allow for sufficiently large energy band-width of positrons at the converter equal to ΔE_s = ±3.5 MeV and about the same energy spread due to the phase dispersion ΔE_φ = ±3.5 MeV corresponding to ±1% energy spread at 500 MeV. Below we will see that using the so called flux concentrator one can largely increase the allowed energy spread at the converter up to e.g. (2, 20) MeV [3] increasing in this way the acceptance of the aperture of positron linac. However we will still have limitation due to the phase lag especially of low energy positrons so that practically the allowed energy spread will not be much larger than (4, 12) MeV.

2.2 Adiabatic Matching. Flux Concentrator

It has been shown above that the QWT has rather narrow positron energy spread acceptance of the order of only few MeV so that its efficiency of positron production can be to low for our purpose. This problem can be solved by using an adiabatically tapered solenoidal magnetic field [3,4]. In such a field a particle orbit encloses a constant magnetic flux so that the quantity P_t^2/B is an adiabatic constant. P_t is the transverse momentum and B the solenoidal magnetic field. Then the following relations are also valid

$$\frac{R_f}{R_i} = \frac{P_{ti}}{P_{tf}} = \sqrt{\frac{B_i}{B_f}} \qquad (12)$$

Here the subscripts i and f correspond to the initial and final values of a given quantity.

The optimum magnetic field shape is obtained by requiring that the field is uniformly adiabatic:

$$\frac{1}{B^2}\frac{dB}{dz} = \text{constant} \qquad (13)$$

Solving for B(z) we obtain:

$$B(z) = \frac{B_i}{1 + b\,z} \tag{14}$$

where

$$b = \left(\frac{B_i}{B_f} - 1\right) L, \quad L \text{ - length of the adiabatic lens.}$$

The changes of the particle orbit radius R(z) are given by:

$$R(z) = R_i \sqrt{1 + bz} \tag{15}$$

R(z) indicates also the changes of the shape of the inner conductor of the flux concentrator to minimize the necessary volume occupied by the field.

According to the SLAC experience [4], to obtain good results by applying the flux concentrator, the following conditions are important:

- The peak value of the magnetic field should be sufficiently high, greater than e.g. 50 kGs.
- The flux concentrator should be as close as possible to the target.
- The flux concentrator should be possible short, to avoid large phase lag of low energy part of positron energy spectrum, remaining always in the adiabatic region of field changes. Usually the length is of the order of 10 cm.

Assume for calculations:

$B_i = B_{max} = 50$ kGs
$B_f = 5$ kGs
Length L = 0.1 m

The adiabatic parameter is then b = 90 m-1 giving the law for the field:

$$B(z) = \frac{B_i}{1 + 90\,z}$$

Radial acceptance is:

$$R_i = R_f \sqrt{\frac{B_f}{B_i}} = 0.9\sqrt{0.1} = 0.28 \text{ cm}$$

Angular acceptance [1]

$$\Phi_{max} = \frac{eR_f}{P}\sqrt{B_i\,B_f} = \frac{1}{\gamma} \quad 586.674\, R_f \sqrt{B_i\,B_f} \tag{16}$$

For the central energy E_c = 8 MeV and the aperture R_f = 0.9 cm we get Φ_{max} = 28.7°.

2.2.1 Energy Acceptance

The SLAC experience has shown that the energy acceptance can be rather large (2, 20) MeV [4] However, the low energy part of positron distribution depends on the capture and acceleration possibilities so as not to loose the positrons because of too large phase lag. In connection with this it seems that the method of first deceleration and then acceleration [5] can improve the efficiency of capture of low energy positrons. Nevertheless the method of using the high gradient capture section directly after the flux concentrator has given also similar range for energy acceptance band [4] It means that using the flux concentrator as a matching device we can surely gather the positrons being in the energy range of (4.5 - 11.5) MeV which at the end of linac will have the total energy dispersion of ±1%.

3. POSITRON FOCUSING ALONG THE LINAC

Considerations presented above were mainly dealing with positron capture and matching between the converter and the linac. Now we will analyse the transport of positrons along the linac. Two types of focusing will be considered : solenoidal and FODO. We begin with equations of motion.

The general equations of motion are

$$\frac{d}{dt}(m\mathbf{v}) = e\,\mathbf{E} + e\,\mathbf{v} \times \mathbf{B} \tag{17}$$

Here, and later on, bold letters denote vectors.

We assume that the particles are already highly relativistic and are moving principally in z- direction so that $\gamma_z = m/m_o \gg 1$,then $v_z = dz/dt = \beta_z c \approx c$ = velocity of light, i.e. $\beta_z \approx 1$ and $\beta_x \approx \beta_y \ll 1$. In this case the longitudinal equation of motion is practically decoupled of the transversal motion and is given by

$$\frac{d}{dt}(m_o\, c\, \gamma\, \beta_z) = e\, E_z \cos\phi \tag{18}$$

It was shown above that for sufficiently high energies (few tens of MeV) the phase ϕ in the case of TW accelerating structures is approaching its asymptotic value equal to $\phi_{as} \approx 0$. Taking into account that $d/dt = (d/dz)(dz/dt) = c\,\beta_z\, d/dz \approx c\, d/dz$ we obtain the equation for energy gain

$$d\gamma/dz = e\, E_z\, /(\, m_o c^2\,) = \alpha \tag{19}$$

For the case of constant gradient (CG) structure $\alpha \approx$ constant and

$$\gamma = \gamma_o + \alpha z \tag{20}$$

Here γ_o corresponds to the initial energy at the entrance to the focusing system and it is assumed that z = 0 at this point.

Since dz = c β dt, than from Eq. (17) we obtain radial equation of motion:

$$(\beta \gamma r')' = \frac{e}{m_o c^2 \beta} (\mathbf{E} + \beta c \mathbf{r'} \mathbf{x} \mathbf{B})_r \tag{21}$$

where prime (‘) denotes differentiation with respect to z.

In absence of external focusing and in the highly relativistic case $\beta_z \approx \beta \approx 1$ defocusing electric and focusing magnetic forces almost cancel, so the right hand side of Eq. (21) can be put equal to zero. Integrating twice we obtain

$$r = r_o + (\beta \gamma r')_o \int_0^z \frac{dz}{\beta\gamma} \tag{22}$$

Here subscript zero denotes arbitrary starting point for which we can accept that R.H.S. of Eq. (21) is zero. Assuming $\beta \approx 1$ and uniform acceleration (e. g. C G structure) $\gamma = \gamma_o + \gamma' z$, we obtain after integration in Eq. (22):

$$r = r_o + r_o' z_o \ln \left(\frac{z + z_o}{z_o} \right) = r_o + r_o' L_a \frac{W_o}{\Delta W} \ln \left(\frac{W_o + \Delta W}{W_o} \right) \tag{23}$$

where

L_a - acceleration length after the point corresponding to the energy
$W_o = \gamma_o m_o c^2$,
$\Delta W = m_o c^2 L_a \gamma'$ is the energy increase on the length
L_a, $z_o = \gamma_o / \gamma'$.

Logarithmic increase of r as a function of z is a consequence of the fact that for $\beta \approx 1$ the transverse momentum remains constant while the longitudinal increases linearly. The quantity $L = z_o \ln(\frac{z + z_o}{z_o})$ can be interpreted as some effective length obtained from the equation

$$r = r_o + r_o' L$$

i.e. the length over which a corresponding radial excursion r in the absence of acceleration will occur [6]. Below we will see that another interpretation is that of contracted length of accelerator as seen from a reference frame moving with the positron.

To have an idea of the radial divergence in DAΦNE positron linac without radial focusing we will calculate the radial increase Δr of the beam radius assuming that the focusing is stopped in some different points along the linac. We will assume for calculations

Initial radius of the beam (m)	$r_o = 0.01$
Accelerating field (MV/m)	$E = 20$
Beam emittance (@ 500 MeV π m rad)	$\varepsilon = 5\ 10^{-6}$

Denoting by W_o the initial energy and by L_a the remaining accelerating length we will have:

$$r_o' = \frac{500\ \varepsilon}{W_o\ r_o}$$

$$z_o = \frac{W_o}{E}$$

$$\Delta r = r_o'\ z_o \ln\left(\frac{z_o + L_a}{z_o}\right)$$

The results for some values of W_o and corresponding L_a are given in Table I.

TABLE I

W_o (MeV)	ΔW (MeV)	L_a (m)	z_o (m)	L (m)	r_o' (rad)	Δr (mm)
100	420	21	5	8.24	$2.5\ 10^{-3}$	20.6
220	300	15	11	9.46	$1.14\ 10^{-3}$	10.75
340	180	9	17	7.22	$7.35\ 10^{-4}$	5.31

It is seen from this Table that even for W_o = 340 MeV the radial excursion will be about 5 mm. It is then evident that the focusing is necessary up to the end of the linac.

For further discussion it is useful to transform Eq. (17) to the system s moving with the accelerated particle using the relation:

$$\gamma_o\ dz = \gamma\ ds \tag{24}$$

REMARK

Solving Eq. (24) for s = f(z) we will obtain:

$$s = \gamma_o \int_0^z \frac{dz}{\gamma} = z_o \ln\left(\frac{z_o + z}{z_o} \right) = L \tag{25}$$

with s = 0 for z = 0 and γ given by Eq. (20). This justify the name of contracted accelerator length used for L given by Eq. (25).

With the aid of relation (24) Eq. (17) can be written in the form:

$$\frac{d^2 \mathbf{r}}{ds^2} = \frac{e}{m_o c^2 \beta} \left(\frac{\gamma}{\gamma_o^2} \mathbf{E} + \frac{\beta c}{\gamma_o} \frac{d\mathbf{r}}{ds} \mathbf{x}\, \mathbf{B} \right) \tag{26}$$

Two types of focusing will be now considered: solenoidal and FODO.

3.1 Solenoidal Focusing

Writing Eq. (26) in cartesian coordinates system (x,y,s) with fields

$$\mathbf{E} = (0,0,E)$$

$$\mathbf{B} = (0,0,B)$$

we obtain for the transversal motion equations

$$\frac{d^2x}{ds^2} - k_1 \frac{dy}{ds} = 0$$

$$\frac{d^2y}{ds^2} + k_1 \frac{dx}{ds} = 0$$

where

$$k_1 = \frac{c\, eB}{m_o c^2 \gamma_o}$$

Differentiating once again both equations and separating the variables we will get for x and y the same equation of the type [7]:

$$\frac{d^3u}{ds^3} + \left(\frac{cB}{z_o E} \right)^2 \frac{du}{ds} = 0 \tag{27}$$

where $zo = \frac{\gamma_o}{\gamma'}$, as defined before.

The main solution to Eq. (27) is:

$$u = A \cos \left(\frac{cB}{z_o E} s \right) \tag{28}$$

Coming back to the laboratory system by expressing s as a function of z with the aid of Eq. (25) one gets:

$$u = A \cos \left(c \frac{B}{E} \log \left(\frac{z_o + z}{z_o} \right) \right) \tag{29}$$

$$\frac{du}{dz} = - A c \frac{B}{E} \frac{z_o}{z_o + z} \sin\left(c \frac{B}{E} \log \left(\frac{z_o + z}{z_o} \right) \right) \tag{30}$$

According to (29) and (30) particles oscillate radially with a constant amplitude and logarithmically increasing wave length whereas the divergence du/dz is decreasing steadilly. However since the amplitude of oscillations remains constant the magnetic field intensity must also remain constant all along the accelerator. It proves that at least from economic point of view the solenoidal focusing is not effective especially for long accelerators.

3.1.1 Magnetic Field in a Solenoid

The magnetic field in a focusing solenoid placed around the accelerating section depends on the bore hole of the accelerating structure and the beam emittance to be transported through it. Assume that the radius of the accelerating structue is R_a then the beam radius R_b should not be greater than $R_b \approx (0.8 \div 0.9)\ R_a$.

We have

$$\rho = R_b/2 = P_t/(eB) \tag{31}$$

where ρ - is the Larmor radius of a particle and P_t is its transversal momentum.

Let the transverse beam emittance for the central energy W_c of captured positrons will be ε_c (π m rad) then we have:

$$\varepsilon_c = R_b\, P_t/Pz \tag{32}$$

$$\rho = R_b/2 = Pt/(eB) = P_z\, \varepsilon_c/R_b$$

and

$$B\, R_b^2 = 2\, \varepsilon_c\, P_z/e = 2 * 1.7045 * 10^{-3}\ \varepsilon_c\, \gamma_c\ (\mathrm{Ts\ m^2}) \tag{33}$$

Or taking into account that $\varepsilon_c = \varepsilon_{510} \dfrac{\gamma_{510}}{\gamma_c}$ we get:

$$B\, R_b^2 = 2 * 1.17045 * 10^{-3}\, \varepsilon_{510}\, \gamma_{510} \qquad (34)$$

Assuming $R_b = 0.8$ cm we obtain

$$B = 2666 \text{ Gs for } \varepsilon_{510} = 5\ 10^{-6}\ \pi \text{ m rad}$$

and

$$B = 5332 \text{ Gs for } \varepsilon_{510} = 1.0\ 10^{-5}\ \pi \text{m rad}$$

For $R_b = 0.9$ cm these values will be correspondingly

$$B = 2238 \text{ Gs and } B = 4476 \text{ Gs.}$$

3.2 Quadrupole Focusing

Let us assume

$$\mathbf{B} = (\, B_x, B_y,\ 0\ ) \qquad (35)$$

With

$$B_x = g\ y,\quad B_y = g\ x,\quad g = g(z) \qquad (36)$$

Inserting (35) and (36) into Eq. (26) and taking into account that $dz/ds = \gamma/\gamma_o$, we obtain both for x and y equation of the form

$$\frac{d^2u}{ds^2} \pm \frac{e}{m_o\ c\ \gamma_o} \frac{\gamma}{\gamma_o}\, g\ u = 0 \qquad (37)$$

For thin lenses further simplification is possible by introducing the quantity Q:

$$Q = \frac{\delta B_y}{dx}\ dz = \frac{\delta B_x}{dy}\ dz = g\ L_{Qo} \qquad (38)$$

Where L_{Qo} is the quadrupole length in the laboratory system. In the moving system we have:

$$L_Q = \frac{\gamma_o}{\gamma}\ L_{Qo}\ ,\ \text{and } g = \frac{Q}{L_{Qo}} = \frac{\gamma_o}{\gamma}\ \frac{Q}{L_Q} \qquad (39)$$

To have a good matching between the beam emittance and channel acceptance we will look for a periodic solution to Eq. (37). We see that in that equation besides u only g and γ depends on s. To obtain a periodic solution as a function of s we should require that

$$\gamma\ g = \gamma_o\ g_o = \text{const.},\ \text{ or }\ g = \frac{\gamma_o}{\gamma}\ g_o\ ,\ \text{and } g_o = \frac{Q}{L_Q} \qquad (40)$$

With the aid of (40) equation (37) can be written in the form [7]

$$\frac{d^2u}{ds^2} \pm k_o\, u = 0 \tag{41}$$

where

$$k_o = \frac{e}{m_o c\, \gamma_o}\quad g_o = \frac{e\, g_o}{p_o}\ ,\ \ p_o = m_o\, c\, \gamma_o \tag{42}$$

According to Eq. (41) the motion in the moving system is that of unaccelerated particle in a periodic quadrupole channel. Such a motion can be realized by keeping the quadrupole strength k_o and length L_{Qo} constant and increasing monotonically distances between quadrupole centers according to Eq. (25) .

3.3 A FODO Focusing System

A FODO system is the most frequently used for positron focusing in long positron accelerators (SLAC, DESY, LEP). However, as it was pointed out in the Introduction the FODO focusing can be applied only when the energy of positrons if sufficiently high of the order of 100 MeV (DESY, LEP). A quick estimation for the value of this energy can be found by comparing the acceptance of a solenoid with that of an alternating gradient system. The acceptance of a solenoid is given by:

$$A_s = \frac{\pi\, a^2\, e\, B}{2\ P} \tag{43}$$

The simplest alternating gradient system is that of alternating singlets the acceptance of which is [6]:

$$A_Q = 0.5\, \pi\, a^2\, Q\, \left(\frac{1 - 0.5\ Q\, l}{1 + 0.5\ Ql}\right)^{1/2} \tag{44}$$

Here and above a is the radius of the transport channel, Ql is the strength of the quadrupol and l is the spacing between the quadrupols. For a fixed spacing l A_Q is maximized for Ql = $\sqrt{5}$ - 1 and the optimum value for A_Q is:

$$A_{Qopt} = 0.300\, \pi\, a^2\, \frac{1}{l}$$

Equating A_s to A_Q and inserting $P = m_o\, c\, \beta\, \gamma$ we obtain for γ :

$$\gamma = \frac{e}{m_o c}\,\frac{B\, l}{0.6} = 586.7\, \frac{B\, l}{0.6}$$

Assuming B = 0.4 T, l ≈ 0.5 m we get γ = 195.6 and the corresponding energy W = 99.4 MeV.

To obtain this energy in DAΦNE linac, the following scheme of positron capture, focusing and acceleration after the e^-e^+ converter can be assumed:

- Adiabatic Flux Concentrator directly after the target.
- (1 ÷ 1.5) m long high gradient capture section increasing the energy by at least W_{cs} = (20 ÷ 30) MeV.
- Two 3 m long CG TW accelerating sections with lower gradient of the order of 12 MeV/m giving the total energy increase of Wsec ≈ 72 MeV. The low gradient has been chosen in order to avoid eventual sparking and breakdown in these sections since they will be immersed in a strong solenoidal magnetic field of (3÷5) kGs which usually increase the probability of ionization and breakdown.

Assuming that the central positron energy at the converter is W_c = 8 MeV, the energy of positrons at the entrance to the FODO focusing will be:

$$Wi = Wc + Wcs + Wsec = 8 + 20 + 72 = 100 \text{ MeV}.$$

Since we have diminished the electric field in the first two sections to 12 MeV/m we should look for a new arrangements of the accelerating sections to fulfil the desired requirements for both high and low current linacs. In relation to the note LC-1 [8] it seems that the best solution will be that of 14, 3 m long accelerating sections plus two ~ 1 m long bunchers and 4 klystrons with the SLED system for RF power supply. The distribution of sections could be as follows: buncher plus 5 sections for high current linac giving 10 + 5*63 = 325 MeV for unloaded electron conversion energy, and capture section plus 9 TW sections for high energy part after the converter. The output energy of this part will be 100 + 7*63 = 541 MeV. The field gradient in the 7 sections with the FODO system will be about 21 MV/m.

The FODO channel is fully described by three parameters: quadrupole strength k, its length $2L_Q$ and the spacing L between the quadrupole centers and may be treated analytically with the use of matrix formalism. In the considered case two of these parameters are already specified or there are at least imposed some constrains which allows for their determination. In fact, as it was mentioned above the constructional requirements due to the necessity of including the accelerating section inside the quadrupole, impose the condition for the minimum length $2L_Q$ of the quadrupole to be larger than (25÷30) cm. Since the minimum distance L between the quadrupoles should be at least equal to the quadrupole length we have for the minimum period of the FODO $L_p = L + 2L_Q \approx (50÷60)$ cm. The changes of the period along the accelerator are then defined by Eq. (25). If L_Q can be kept constant it remains only to find the quadrupoles strength k so as to optimize the matching of the beam emittance to the channel acceptance. To find this matching we have followed the method given in [7,9].

The following formulae were used for these considerations:

$$\varphi = l\sqrt{k} \tag{45}$$

$$m^2 = \frac{1 + \tanh\varphi(\tan\varphi + \frac{L}{l}\varphi)}{1 - \tan\varphi(\tanh\varphi + \frac{L}{l}\varphi)} \tag{46}$$

$$\varepsilon^2 = \frac{R^4\,\varphi^2}{l^2}\;\frac{m^2}{(1 + m^2)}\;\frac{\tan\varphi(\mathrm{ctanh}\varphi + \frac{L}{l}\varphi) - 1}{1 + \mathrm{ctanh}\varphi(\tan\varphi + \frac{L}{l}\varphi)} \tag{47}$$

where

k	= lens strength.
$2l$	= lens length.
L	= distance between lenses. Period is then $L + 2l$.
m	= E_{max}/E_{min} - beat factor.
E_{max}	= maximum of the envelope in the middle of the foc. quad.
E_{min}	= minimum of the env. in the middle of the defoc. quad.
ε	= acceptance of the FODO channel.
R	= minimum radius of the accelerating structure.

Analytical cosiderations by Steffen [9] have shown that the maximum of channel acceptance is found for some value of $m > 1$. To find the optimum value of the beat factor m and the corresponding strengths of quadrupoles lenses a numerical program has been written. It is composed basically of two parts. In the first part the optimum parameters of the FODO channel are found, in the second the particle envelopes in both planes x and y are calculated using these optimum parameters.

The calculation of the FODO parameters begins with the definition of the initial values for energy, energy gradient, the length $2l$ of the quadrupoles and the first spacing L_o between the quadrupoles. The first period of the FODO channel is then $L_1 = L_o + 2l$ in the laboratory system and:

$$s_1 = z_o \ln\left(1 + \frac{L_1}{z_o} \right) \tag{48}$$

in the system moving with the particle. Since the motion in the moving system is periodic then s_1 is the period of oscillations in this system.The value of s for the nth period is $s_n = n\, s_1$ and the corresponding length in the laboratory system is:

$$z_n = z_o \left(\exp\left(\frac{s_n}{z_o} \right) - 1 \right) = z_o \left(\exp\left(\frac{nL_1}{z_o}\right) - 1 \right) \tag{49}$$

Here z_n is the distance of the center of the nth quadrupole from the beginning i.e. from s = z = 0. The length of the nth period is then:

$$L_n = z_n - z_{n-1} = L_1 \exp\left(\frac{(n-1)\, L_1}{z_o} \right) \qquad (50)$$

Knowing the accelerator length L_a putting it equal to z_n and solving for n we obtain the number of quadrupoles. Assuming some value for the beat factor m and solving Eq. (44) for ϕ (n) we find the quadrupole strength all along the accelerator. Inserting ϕ into Eq. (45) the acceptance of the channel is found and can be compared with the corresponding beam emittance.

In the second part the equation of the envelope E:

$$\frac{d^2E}{dz^2} \pm k\, E - \frac{\varepsilon^2}{E^3} = 0 \qquad (51)$$

is solved using the found values for k(n) and corresponding values of beam emittance ε(n).

4. *Numerical calculations*

According to the above considerations a FODO focusing can be applied for the last 7 accelerating sections. The initial parameters of the system can be chosen close to the following:

Quadrupol length	$2\,l$	= 0.3 m
The first distance between quadrupoles	L_o	= 0.3 m
Initial energy	W_o	= 100 MeV
Average accelerating gradient	E	= 21 MV/m
Effective acceleration length	L_a	= 21 m
Useful iris radius of acceleration section	R	= 0.9 cm
Beam emittance (@ 100 MeV π m rad)	ε	= $2.5\ 10^{-5}$

We have then:

The length of the first period	L_1	$= L_o + 2l = 0.6$ m
Parameter	z_o	$= \gamma_o/\gamma' \approx 4.8$ m
Minimum number of quadrupoles		

$$N_{qmin} \geq \frac{z_o}{L_1} \ln\left(1 + \frac{L_a}{z_o} \right) = 13.4$$

The minimum number of quadrupoles should be then at least 14. However, it is rather sure that it will greater since to calculate N_{qmin} we have assumed that the linac length is equal only to its active part L_a = 21 m. In practice this length must be greater so that both L_a and z_o should be increased. In order to take this into account the further calculations will be increased. In order to take this into account the further calculations will be made with $L_a \approx 23$ m and the average gradient E=20 MV/m.

The beam parameters at the exit of the solenoidal focusing have axial symmetry so before injection into the FODO focusing there should be a matching transition between this two systems transforming the equal amplitudes in two perpendicular planes into the waist with a beat factor m > 1. In our case we have chosen m ≈ 2.

The parameters of the matching transition were found with the aid of the modified LEDA [9] program (modifications consisted in including changes of parameters due to acceleration in the linac.) The system is composed of the quadrupole triplet followed by two and half quadrupole singlets. The triplet in which the envelope amplitude can be the highest is placed between the sections, the other quadrupoles , or some of them, can be wrapped-around the sections. The second half of the last quadrupol belongs already to the regular FODO system. The arrangement of the matching transition together with the beam envelops are presented in Fig. 1 and the parameters of the system are given in Table II.

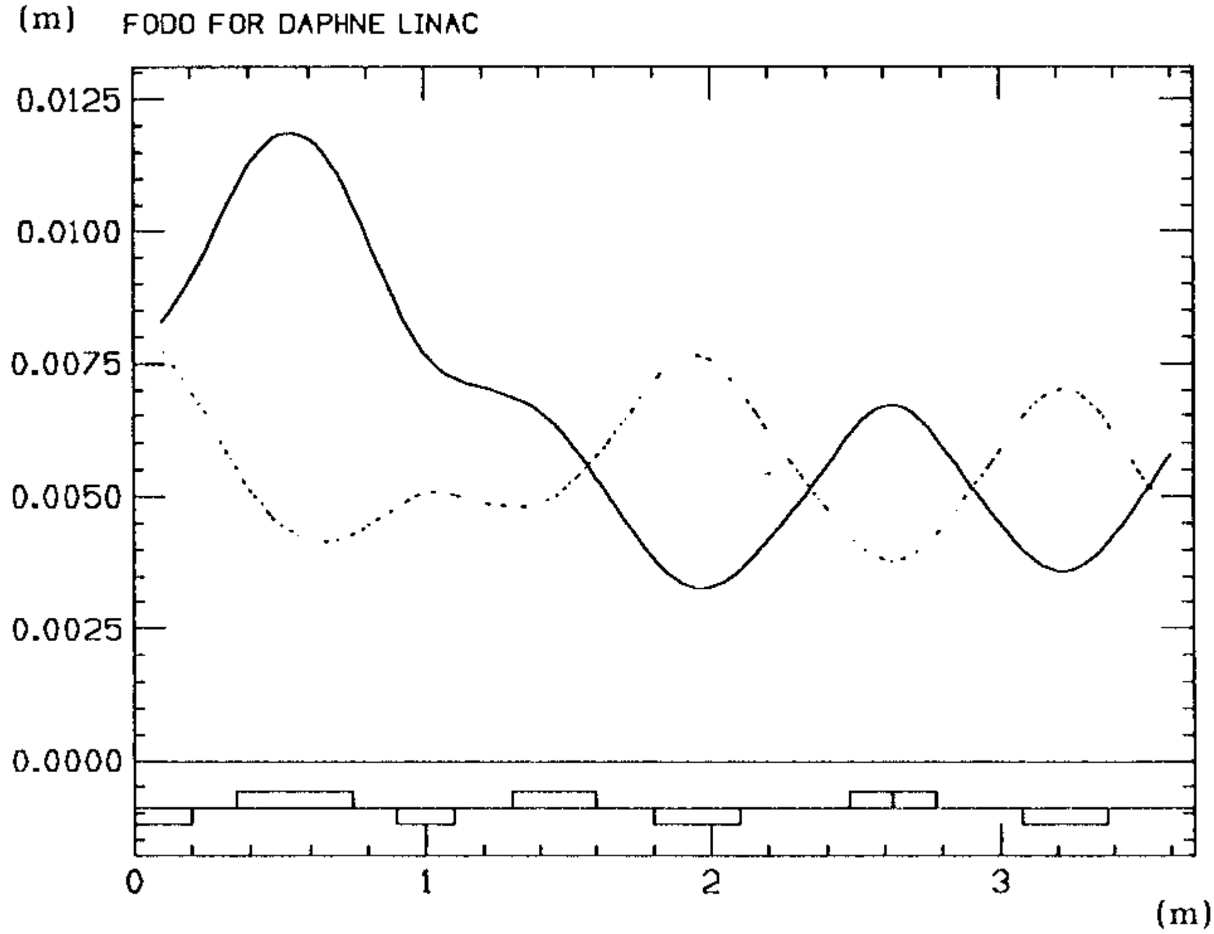


Fig. 1 *- Matching system between the solenoidal and FODO focusing for DAΦNE positron Linac. R_x: continuos line; R_z: dashed like. Both scales are in meters.*

Table II. Parameters of matching transition between the solenoidal and FODO focusing for DAΦNE positron LINAC

ENERGY (MeV)	100.0
ELECTRIC FIELD (MeV/m)	0.100E-02
EMM.X (PI m rad)	0.2500E-04
EMM.Y (PI m rad)	0.2500E-04
TOTAL LENGTH (m)	3.68

INITIAL VALUES

R_{X0} (m)	R_{Z0} (m)	alfax$_0$	alfaz$_0$
0.8000D-02	0.8000D-02	0.0000D+00	0.0000D+00

ACHIEVED CONVERGENCE = 0.708542002D-08

TRANSPORT-TYPE SOLUTION

	TYPE	LENGTH (m)	GRAD. (T/m)
1	QD	0.200000000D+00	0.219999976D+01
2	D1	0.150000000D+00	0.000000000D+00
3	QF	0.400000000D+00	0.175562656D+01
4	D1	0.150000000D+00	0.000000000D+00
5	QD	0.200000000D+00	0.219999976D+01
6	D1	0.200000000D+00	0.000000000D+00
7	QF	0.300000000D+00	0.129672261D+01
8	D1	0.200000000D+00	0.000000000D+00
9	QD	0.300000000D+00	0.233370507D+01
10	D1	0.376015009D+00	0.000000000D+00
11	QF	0.150000000D+00	0.273600177D+01
12	D1	0.500000000D-03	0.000000000D+00
13	D1	0.500000000D-03	0.000000000D+00
14	QF	0.150000000D+00	0.273600177D+01
15	D1	0.300000000D+00	0.000000000D+00
16	QD	0.300000000D+00	0.271200175D+01
17	D1	0.300000000D+00	0.000000000D+00

R_{min} (m)	R_{max} (m)
0.3266E-02	0.1186E-01

Starting from the end of the matching system (middle of the third singlet) a regular FODO system was calculated with the aid of the program FODO and using the above described procedure. The length of the ith period was calculated with the aid of equation (50) and then the lens strength k was found by solving the equations (45), (46) and (47) with assumed value of the beat factor m.

At last the beam envelope has been found by solving the equation (51) with the intial parameters corresponding to the waist at the beginning of the FODO system and given in Table II i.e.:

$$Rx = 0.0067 \text{ m}, \quad Rz = 0.0038 \text{ m}, \quad alfax = alfaz = 0.$$

The numerical calculations have shown that sufficiently good solution is obtained if the beat factor m changes uniformly in the range m = (1.9÷2.5) along the system composed of 15 quadrupols. It was also assumed that the energy changes uniformly with a gradient of 20 MV/m starting from 100 MeV. The results of these calculations are presented in Table III and in Fig. 2.

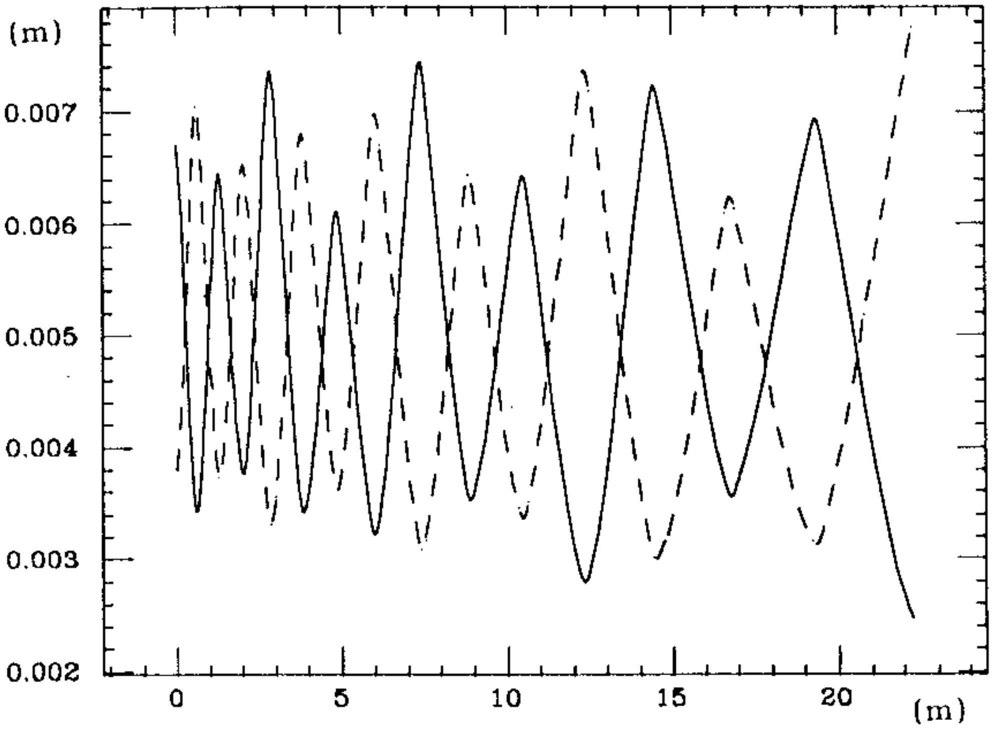
Table III. Parameters of the FODO system for the DAΦNE positron LINAC.

I	LC(I)	LN(I)	EMITT	ADMIT	G (T/M)	KQ(M-2)
1	0.6000E+00	0.6000E+00	0.2500E-04	0.1102E-03	0.2736E+01	0.8159E+01
2	0.6716E+00	0.1272E+01	0.2232E-04	0.1094E-03	0.2712E+01	0.7225E+01
3	0.7518E+00	0.2023E+01	0.1993E-04	0.1071E-03	0.2696E+01	0.6418E+01
4	0.8416E+00	0.2865E+01	0.1780E-04	0.1038E-03	0.2688E+01	0.5715E+01
5	0.9421E+00	0.3807E+01	0.1589E-04	0.9996E-04	0.2684E+01	0.5099E+01
6	0.1055E+01	0.4862E+01	0.1419E-04	0.9580E-04	0.2685E+01	0.4557E+01
7	0.1180E+01	0.6042E+01	0.1268E-04	0.9149E-04	0.2690E+01	0.4078E+01
8	0.1321E+01	0.7363E+01	0.1132E-04	0.8714E-04	0.2697E+01	0.3652E+01
9	0.1479E+01	0.8843E+01	0.1011E-04	0.8281E-04	0.2706E+01	0.3274E+01
10	0.1656E+01	0.1050E+02	0.9030E-05	0.7856E-04	0.2717E+01	0.2937E+01
11	0.1853E+01	0.1235E+02	0.8065E-05	0.7442E-04	0.2730E+01	0.2636E+01
12	0.2075E+01	0.1443E+02	0.7204E-05	0.7041E-04	0.2743E+01	0.2366E+01
13	0.2322E+01	0.1675E+02	0.6435E-05	0.6656E-04	0.2758E+01	0.2125E+01
14	0.2600E+01	0.1935E+02	0.5747E-05	0.6286E-04	0.2773E+01	0.1909E+01
15	0.2910E+01	0.2226E+02	0.5134E-05	0.5932E-04	0.2789E+01	0.1715E+01

where:

LC(I) - length of the FODO 'period' (m)
LN(I) - length of the FODO system along the linac (m)
EMITT. - Beam emittance (m rad)
ADMIT - FODO acceptance (m rad)
KQ - Lens strength (m^{-2})

Using the calculated parameters of the FODO system the beam envelopes were also found with the aid of the modified LEDA program. The results are presented in Fig. 3 and in Table IV.



FODO for DAFNE positron LINAC R_X : continuos R_Z : dshed line

Fig. 2 *- Envelopes R_X and R_Z as calculated by the program FODO.*

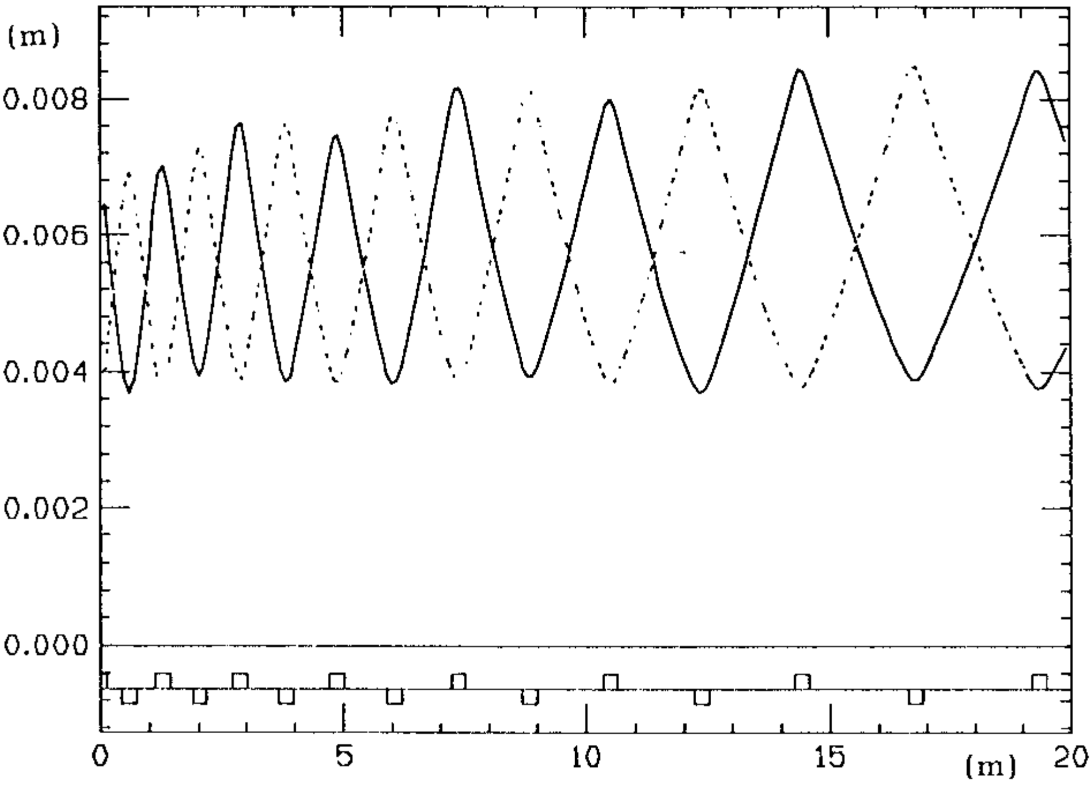


Fig. 3 *- Envelopes R_X and R_Z as calculated by the program LEDA using the FODO parameters calculated by the program FODO - R_X : continuos line; R_Z : dashed line.*

Table IV. FODO FOR DAΦNE LINAC.
Results obtained with the aid of the program LEDA

INITIAL VALUES

R_{X0} (m)	R_{Z0} (m)	$alfax_0$	$alfaz_0$
0.670D-02	0.380D-02	0.000D+00	0.000D+00

ENERGY (MeV)	100.0
ELECTRIC FIELD (MeV/m)	20.0
EMM.X (π m rad)	0.2500E-04
EMM.Y (πm rad)	0.2500E-04
TOTAL LENGTH (m)	20.0

TRANSPORT-TYPE SOLUTION

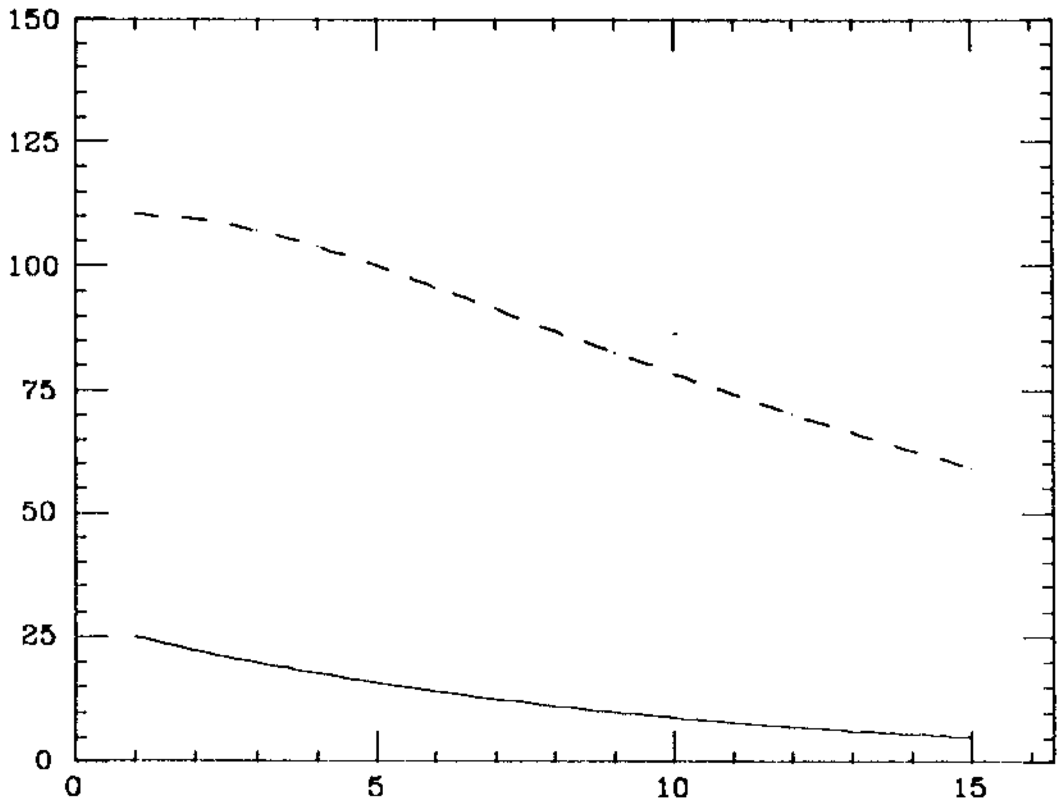
	TYPE	LENGTH (m)	GRAD. (T/m)
1	D1	0.500000000D-03	0.000000000D+00
2	QF	0.150000000D+00	0.273599970D+01
3	D1	0.300000000D+00	0.000000000D+00
4	QD	0.300000000D+00	0.271199970D+01
5	D1	0.371600000D+00	0.000000000D+00
6	QF	0.300000000D+00	0.269599970D+01
7	D1	0.451800000D+00	0.000000000D+00
8	QD	0.300000000D+00	0.268799970D+01
9	D1	0.541600000D+00	0.000000000D+00
10	QF	0.300000000D+00	0.268399970D+01
11	D1	0.642100000D+00	0.000000000D+00
12	QD	0.300000000D+00	0.268499970D+01
13	D1	0.755000000D+00	0.000000000D+00
14	QF	0.300000000D+00	0.268999970D+01
15	D1	0.880000000D+00	0.000000000D+00
16	QD	0.300000000D+00	0.269699970D+01
17	D1	0.102100000D+01	0.000000000D+00
18	QF	0.300000000D+00	0.270599970D+01
19	D1	0.117900000D+01	0.000000000D+00
20	QD	0.300000000D+00	0.271699970D+01
21	D1	0.135600000D+01	0.000000000D+00
22	QF	0.300000000D+00	0.272999970D+01
23	D1	0.155300000D+01	0.000000000D+00
24	QD	0.300000000D+00	0.274299970D+01
25	D1	0.177500000D+01	0.000000000D+00
26	QF	0.300000000D+00	0.275799969D+01
27	D1	0.202200000D+01	0.000000000D+00
28	QD	0.300000000D+00	0.277299969D+01
29	D1	0.230000000D+01	0.000000000D+00
30	QF	0.300000000D+00	0.278899969D+01
31	D1	0.500000000D+00	0.000000000D+00

R_{min} (m)	R_{max} (m)
0.3692E-02	0.8489E-02

The comparison of Figs. 2 and 3 shows that the general behaviour of the beam envelopes obtained with the aid of these two different programs is similar. In both cases there is a slow increase of the "average" envelopes amplitudes. In the case of the LEDA calculations this increase is practically monotonic and slightly faster: the amplitude changes from ~ 6.7 mm to about 8.5 mm, in another there are some oscillations of the envelopes but on the average they also increase to ~ 7.5 mm.

The quadrupole magnetic field gradient is almost constant along the system: it decreases slightly at the beginning from 2.736 T/m to 2.684 T/m in the fifth quadrupol and rises monotonically to 2.789 T/m at the end. The magnetic lens strength k diminishes monotonically from 8.16 m^{-2} to 1.715 m^{-2} along the system.

Interesting is the comparison between the FODO acceptance and beam emittance given in Fig. 4: the admittance decreases monotonically between (1.1 ÷ 0.59) 10^{-4} π m rad whereas the beam emittance changes within the limits (2.5 ÷ 0.51) 10^{-5} π m rad. It means that surely the total beam phase space passes through the channel, providing that envelopes amplitudes are smaller than the channel radius.

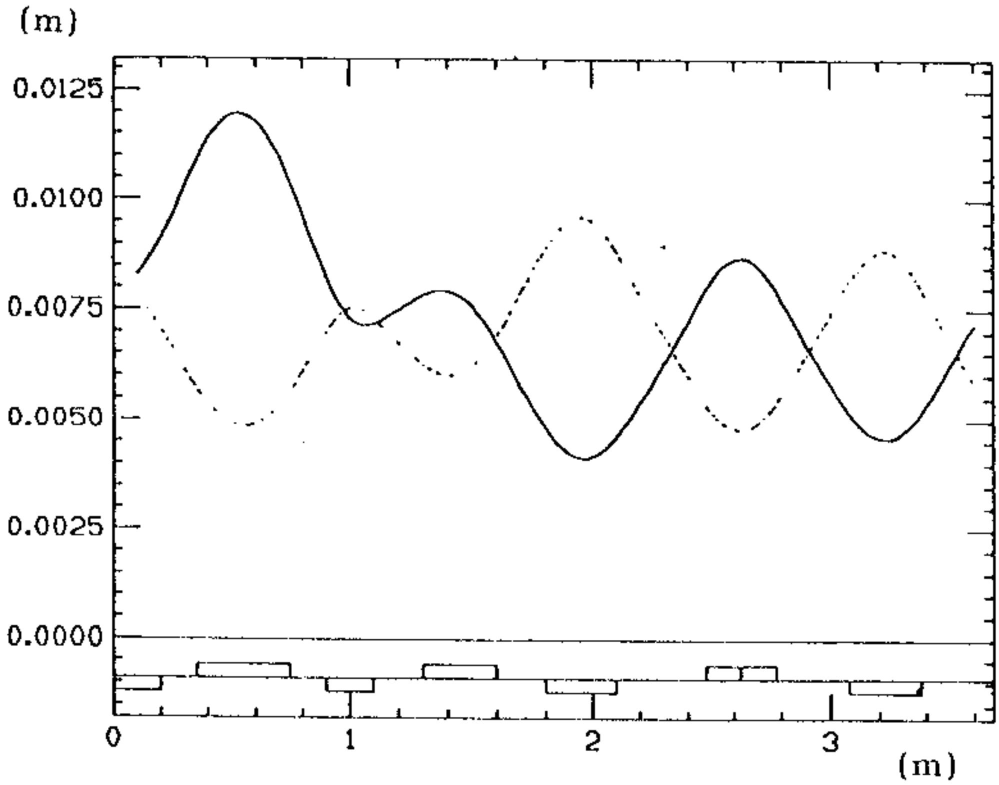


Quadrupole Number   Continuos Line : Beam Emittance   Dashed Line : FODO Acceptance

Fig. 4 *- Beam emittance and FODO acceptance of the DAΦNE positron Linac.*

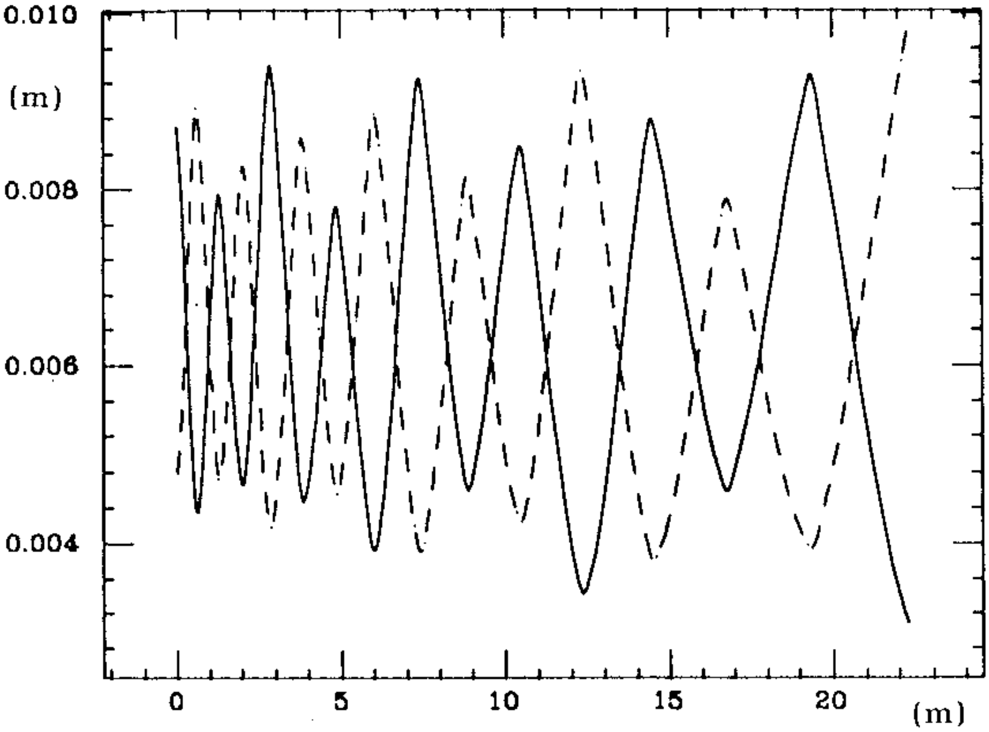
A preliminary calculation of the beam transport with emittance of $\varepsilon = 4\ 10^{-4}\ \pi$ mrad (@ 100 MeV) corresponding to $8\ 10^{-5}\ \pi$ m rad (@ 500 MeV) have been also done. It was possible to keep the same gradients in the regular FODO lattice and to change only the gradients in the matching section of the transport line. The results of calculations are shown in Figs. 5, 6 and 7. In Fig. 5 the matching part of the beam transport transport is presented. Figs. 6 and 7 show the envelopes in the FODO system as calculated by the programs FODO and LEDA correspondingly. Due to the higher emittance the amplitude of the envelop can increase up to about 9.5 mm according to the FODO results and up to almost 11 mm according to the LEDA calculations. Since the minimum diameter of the accelerating structure is ~ 19 mm we can have some beam losses even if again the admittance of the FODO is larger than the beam emittance. The possible solution can be to use higher magnetic field in the solenoid and arrive with smaller radius at the beginning of the FODO latice.

The simulations have been also done for the transport of electrons for which it was assumed the emittance $\varepsilon = 5.\pi\, 10^{-6}$ m rad and correspondingly smaller radius of 2 mm at the end of the solenoidal focusing. Again it was assumed that the parameters of the FODO lattice should remain unchanged and also the position of the elements of the matching system. The results of the calculations are presented in Figs. 8, 9, and 10. It is seen from these figures that it will be much easier to transport the beam of electrons since it has smaller emittance.



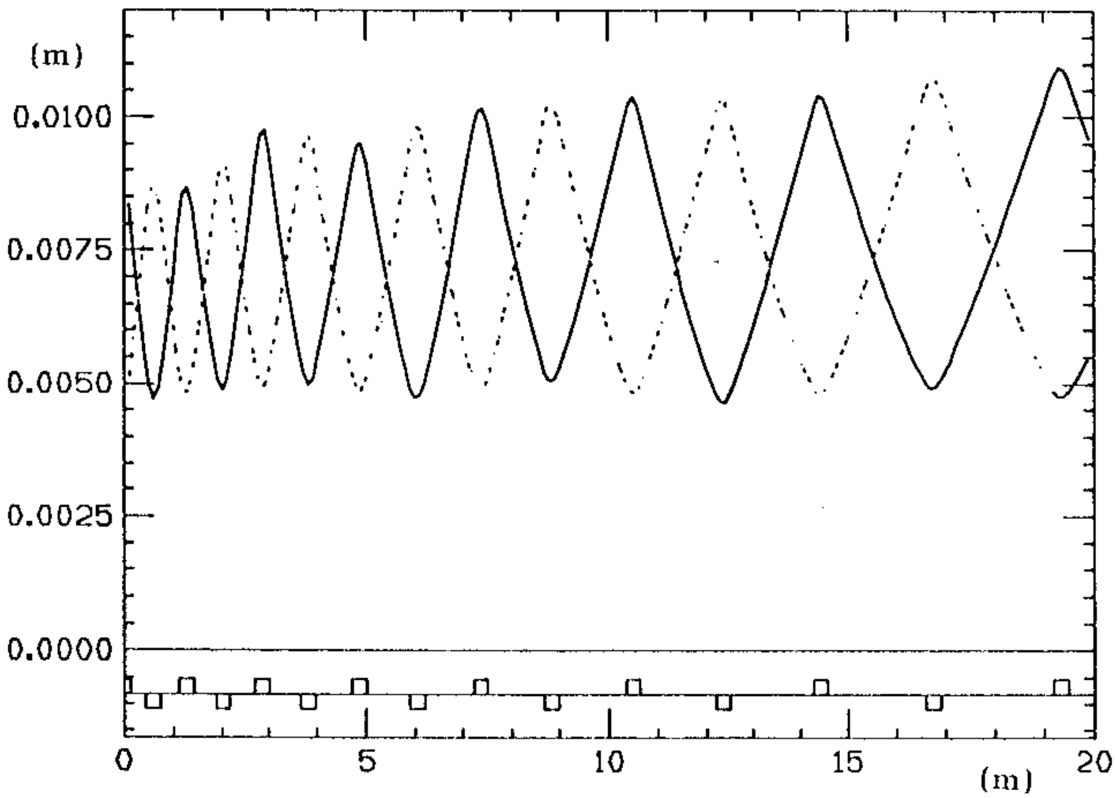
Emittance $\varepsilon = 4\pi 10^{-5}$ (@100 MeV) R_x : continuous line; R_z : dashed line

Fig. 5 *- Matching transition for the DAΦNE FODO focusing.*



Emittance $\varepsilon = 4\pi 10^{-5}$ *m rad, beat factor m = (1.9 ÷ 2.5)* R_X *: continuous line;* R_Z *: dashed line*

Fig. 6- *Envelopes* R_X *and* R_Z *calculated by the program FODO*



Emittance $\varepsilon = 4\pi 10^{-10}$ *m rad, beat facto m = (1.9 ÷ 2.5)* R_X *: continuous line;* R_Z *: dashed line*

Fig. 7 - *Envelopes* R_X *and* R_Z *calculated by the program LEDA*

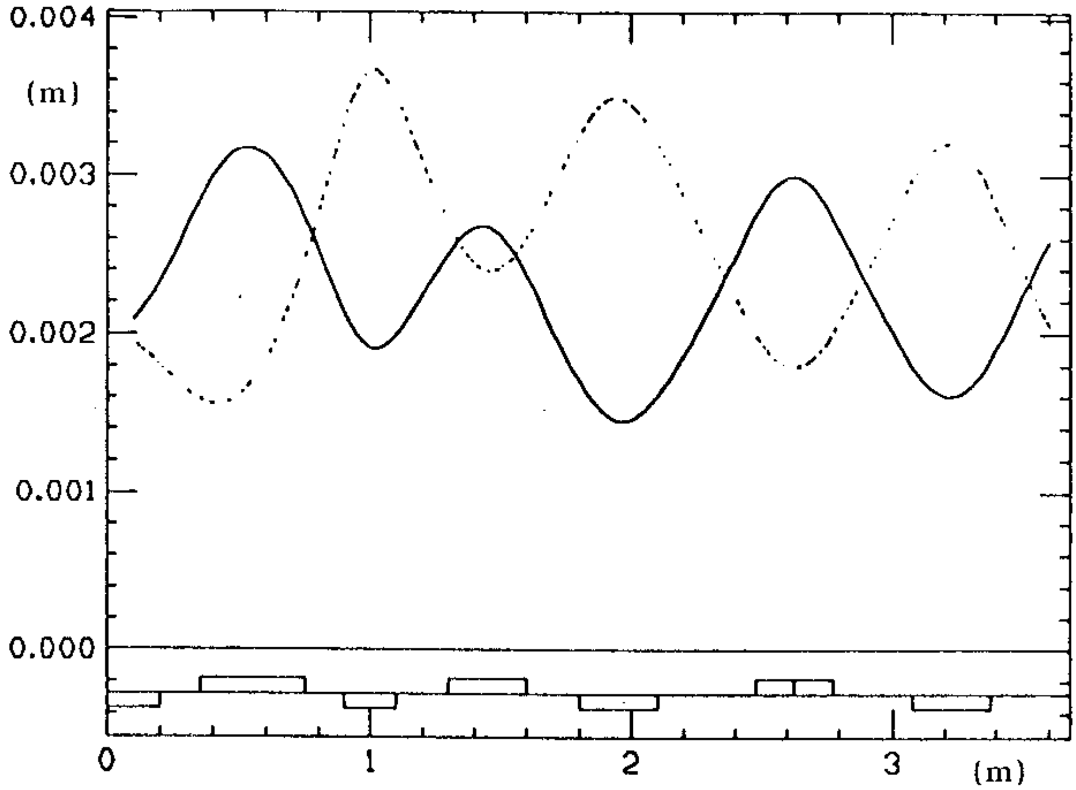
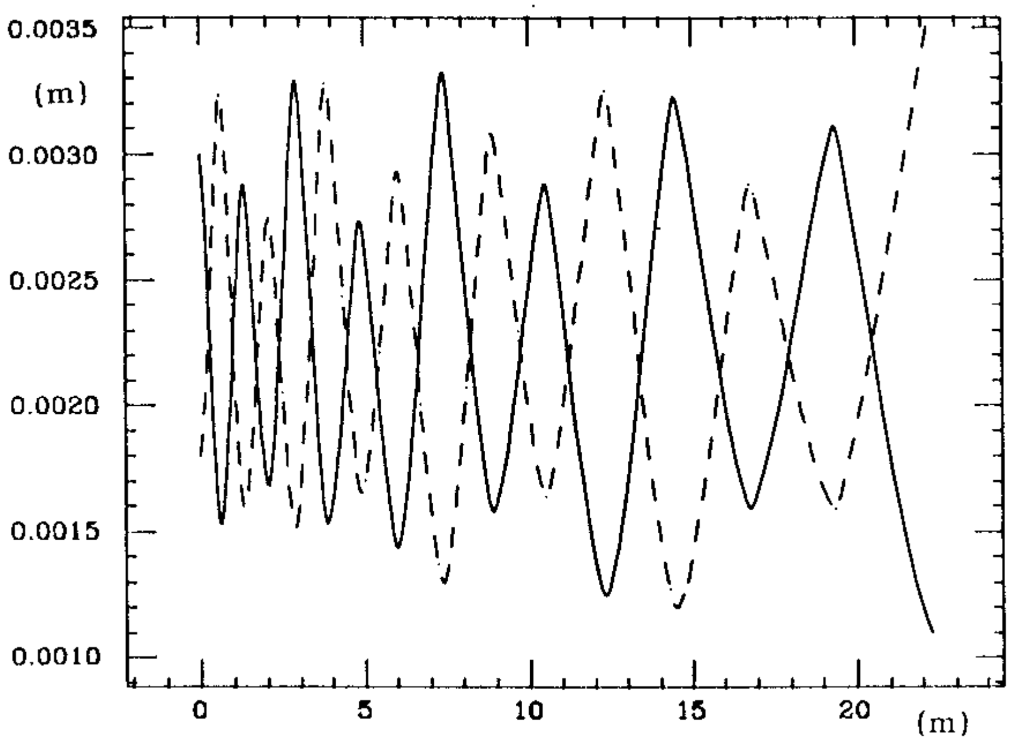
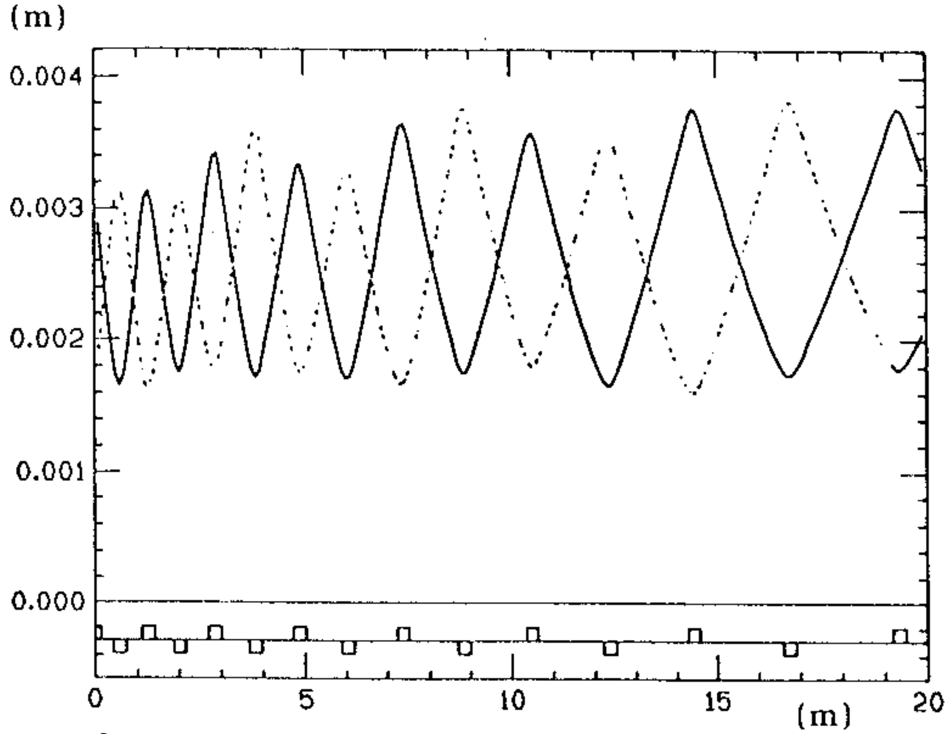


Fig. 8 *- Matching transition for DAΦNE FODO focusing in the case of electrons* $\varepsilon = 5\pi 10^{-6}$ *m rad (@ 100 MeV) -* R_X *: continuous line;* R_Z *: dashed line*



Electrons $\varepsilon = 5\pi 10^{-6}$ *m rad, beat facto m = (1.9 ÷ 2.5)* R_X *: continuous line;* R_Z *: dashed line*

Fig. 9 *- Envelopes* R_X *and* R_Z *calculated by the program FODO*



Electrons $\varepsilon = 5\pi 10^{-6}$ *m rad, beat facto m = (1.9 ÷ 2.5)* R_X *: continuous line;* R_Z *: dashed line*

Fig. 10 *- Envelopes* R_X *and* R_Z *calculated by the program LEDA*

It should be stressed, however, that all these results are preliminary and made only for the " ideal " system without the errors. A much more work should be still done when the linac structure will be decided so that all real parameters of the focusing system can be taken into account together with the admissible tolerances.

There was rather good agreement between the results obtained with programs LEDA and FODO. Since the program FODO was made specially for linacs it is simpler and faster than LEDA and it can be used for a quick calculation of the FODO focusing parameters and the beam behaviour estimation.

REFERENCES

[1] R. Chechab, A Positron Beam For The Linear Collider Scheme of a B- Factory. LAL - Report. Orsay Nov. 1987.
[2] J. Haissinski, Focusing Devices For a Positron Beam at The Linear Accelerator of Orsay. NIM 51, pp.181-196, 1967.
[3] D. Sherden, Single Pass Collider MEMO CN-22 SLAC 5.07. 1980.
[4] K. Moffeit, Positron Source: first 50 nsec. SLAC Single Pass Collider MEMO 268.
[5] M. James et al., A New Target Design SLAC Pub. 5215 , April 1990 (A/E).
[6] R.H.Helm et al., The Stanford Two Mile Accelerator. Chapter 7. Beam Dynamics. W.A Benjamin 1968.
[7] H. Wiedemann, A Positron Focusing System For Linear Accelerators. DESY Rep. H14, Hamburg Nov 1966.
[8] S. Kulinski, R. Boni, B. Spataro, M. Vescovi, G. Vignola, The Linear Accelerator for DAΦNE Injection System. DAΦNE Technical Note LC-1 , Frascati Feb. 1991.
[9] K.G. Steffen, High Energy Beam Optics. Interscience 1965.
[10] G. Vignola, Program LEDA.