DAΦNE TECHNICAL NOTE

INFN - LNF, Accelerator Division

Frascati, May 22, 1995

Note: **G-33**

ANALYTICAL FORMULAE FOR MAGNETIC MULTIPOLES

M. Bassetti, C. Biscari

CONTENTS

INTRODUCTION

Magnetic multipole elements play an important role in particle accelerators. The main purpose of this paper is to obtain accurate values of the magnetic field (satisfying Maxwell equations exactly), which can be used in theoretical computations. For real dipoles and quadrupoles, made of iron and coils, this is almost impossible because of construction errors, limited accuracy in magnetic design codes, and, last but not least, computing time. For that reason we decided to take into account only magnetic fields created by currents in vacuum.

We started with numerical calculations similarly to ref. [1,2,3]. Being our work purely speculative, our current distributions were not linked to practical realization. Thanks to this conceptual freedom we discovered an interesting property. We simulated each pole of a quadrupole (see Fig. 1) with a series of rectangular coils with the currents along the longitudinal sides distributed according to

$$I(\theta) = I_c \cos 2\theta$$

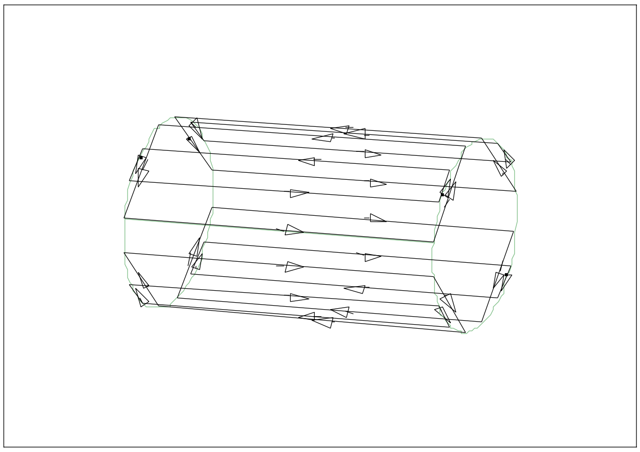

Figure 1 - Coil network generating quadrupolar field plus odd harmonics

Analyzing the generated field we found as expected the presence of the odd harmonic terms starting from the dodecapole. These terms did not change meaningfully distributing the current of each pole on more coils and therefore we attributed the higher harmonics to the coil shape. In fact the higher harmonics disappeared changing the leaning of the head currents from the chords of the end circumferences to their arcs (see Fig. 2).

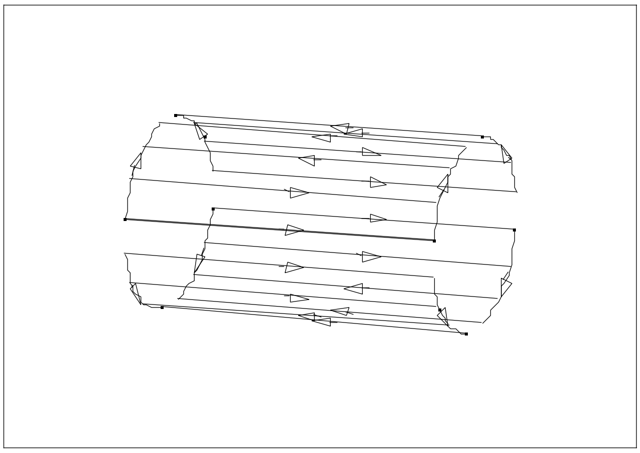

Figure 2 - Coil network generating pure quadrupolar field

Subtracting the currents of Fig. 1 from those of Fig. 2, the currents leaning on the longitudinal sides disappear and the result are coils of currents leaning only on the cylinder basis and each of them made up of a chord and the corresponding piece of arc (see Fig. 3). This arrangement of currents neutralizes the dodecapole and successive terms. We cannot avoid thinking that the usual way of neutralizing the unwanted dodecapole of a normal quadrupole with chamfers on the quadrupole ends and the previous modification of the current arrangement in our model are equivalent.

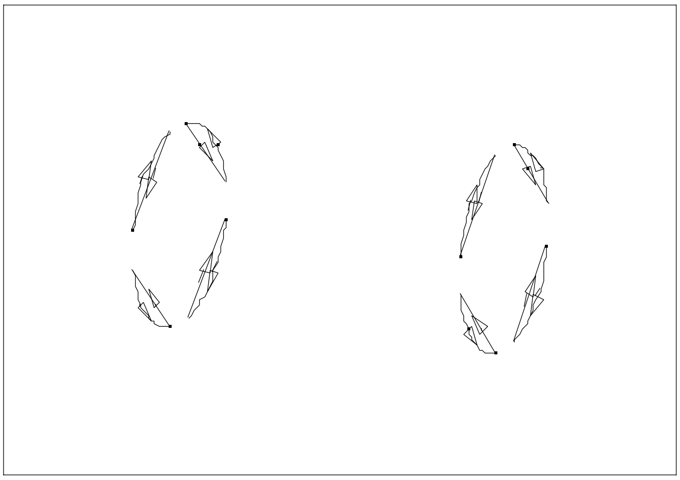

Figure 3 - Coil network generating high harmonic terms

Later on we discovered that these current arrangements are mathematically manageable. The conceptual path from the numerical to the analytical approach was very exciting. This paper will discuss only the analytical one. It will be shown that there exists a mathematical treatment which holds for rectangular dipoles, quadrupoles, sextupoles, etc.

1) POTENTIAL AND FIELDS OF MULTIPOLES

1.1) Magnetic Potential

In the current-free region the magnetic field components of a multipole magnet can be obtained as gradient of a scalar potential. Different mathematical approaches can be used to determine the potential. We will do large use of the formulae developed in [1,2,3,4]. For the developments in the following of this paper we report in this paragraph the most important. Using cylindrical coordinates the potential corresponding to the harmonic *m* can be written as:

$$P_m(r,\phi,z) = \frac{r^m \sin m\phi}{m!} G_m(r,z) \qquad 1.1.1)$$

Replacing the factor $\sin m\phi$ by $\cos m\phi$ the potential represents skew elements. In the solenoidal case $(m=0)$ only the cosine choice is meaningful and the potential coincides directly with $G_m(r,z)$ (see paragraph 3). For the other cases, assuming:

$$G_m(r,z) = \sum_{p=0}^{\infty} G_{m2p}(z)\, r^{2p} \qquad 1.1.2)$$

the different terms G_{m2p} $(p>0)$ are related to the function G_{m0} by the formula:

$$G_{m2p}(z) = (-1)^p \frac{m!}{4^p(m+p)!p!} \frac{d^{2p}G_{m0}}{dz^{2p}} \qquad 1.1.3)$$

and the potential for a single harmonic becomes:

$$P_m(r,\phi,z) = \frac{r^m \sin m\phi}{m!} \left\{G_{m0}(z) + G_{m2}(z)\, r^2 + G_{m4}(z)\, r^4 + ...\right\} \qquad 1.1.4)$$

Let us emphasize that:

i) The potential is a *2D* potential times a function $G(r,z)$, and the dependence on ϕ is the same of the *2D* solution.

ii) Once fixed m, all the potential is deducible from the function $G_{m0}(z)$.

iii) The order of the harmonic m is also the power of r. This assures that if a potential can be written as:

$$P_m(r,\phi,z) = \sin m\phi\; Q(r,z) \qquad 1.1.5)$$

the lowest power of r is also m, namely:

$$Q(r,z) = r^m \sum_{p=0}^{\infty} G_{m2p}(z)\, r^{2p} \qquad 1.1.6)$$

The potential in eq. 1.1.4) can be differentiated m times with respect to r at the point $r=0$. The result, which will be useful in the following to deduce $G_{m0}(z)$, is:

$$\sin m\phi\; G_{m0}(z) = \left(\frac{\partial^m P_m(r,\phi,z)}{\partial r^m}\right)_{r=0} \qquad 1.1.7)$$

1.2) Magnetic Field

Since the scalar magnetic potential is just a mathematical tool we define the magnetic field simply as its gradient, without the usual negative sign. The three field components in cylindrical coordinates are hence:

$$B_r(r,\phi,z) = \frac{\sin m\phi}{m!} \sum_{p=0}^{\infty} (m+2p)\; G_{m2p}(z)\, r^{2p+m-1} \qquad 1.2.1)$$

$$B_\phi(r,\phi,z) = \frac{\cos m\phi}{(m-1)!} \sum_{p=0}^{\infty} G_{m2p}(z)\, r^{2p+m-1} \qquad 1.2.2)$$

$$B_z(r,\phi,z) = \frac{\sin m\phi}{m!} \sum_{p=0}^{\infty} G_{m2p+1}(z)\, r^{2p+m} \qquad 1.2.3)$$

where writing $G_{m2p+1}(z)$ we mean the derivative with respect to z of $G_{m2p}(z)$:

$$G_{m2p+1}(z) = \frac{dG_{m2p}(z)}{dz} \qquad 1.2.4)$$

From the transverse cylindrical components the cartesian ones are simply obtained:

$$B_x = B_r \cos\phi - B_\phi \sin\phi \qquad 1.2.5)$$

$$B_y = B_r \sin\phi + B_\phi \cos\phi \qquad 1.2.6)$$

From the point of view of the magnetic field the adoption of a scalar or a vector potential are equivalent. Deduction of 1.2.1/1.2.3 from a vector potential are known since 1966 [5,6]. We discovered these two references in [7] where they are applied to understand the effect of the fringing fields of the insertion quadrupoles in DORIS.

2) CURRENT DISTRIBUTIONS

2.1) Current Distributions Generating Pure Multipole Potentials

While in refs. 2, 3 the purpose is the computation of the potential starting from fixed current distributions, in this paper we face the following question: do a current distribution in vacuum exist such to realize a pure multipolar potential?

The answer is positive, as already mentioned in the introduction. Figure 4 shows a cylindrical geometry of a current distribution which realizes a pure multipole of order m. The current $I(\theta)$ circulating on the two circumferences of the bases is:

$$I(\theta) = I_c \sin m\theta \qquad 2.1.1)$$

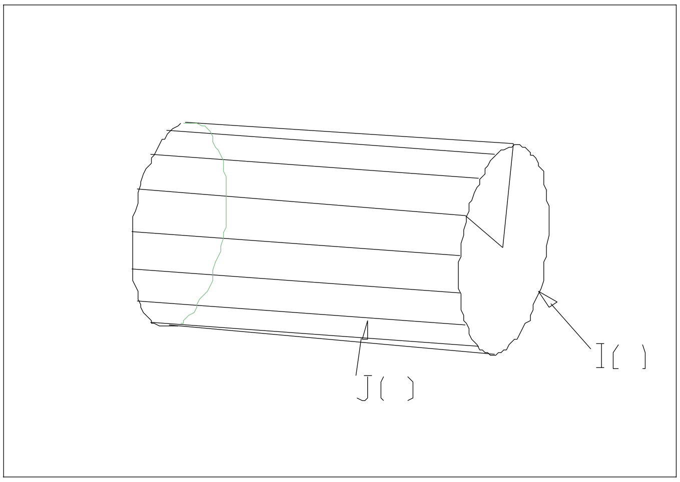

Figure 4 - Current distribution generating a pure multipole

On the lateral surface of the cylinder the linear current density J (A m^{-1}) must be such that:

$$J(\theta)\, Rd\theta = I(\theta+d\theta) - I(\theta) \qquad 2.1.2)$$

from which:

$$J(\theta) = \frac{1}{R}\frac{dI(\theta)}{d\theta} = \frac{mI_c}{R} \cos m\theta \qquad 2.1.3)$$

To simplify the problem we consider a network made of a finite number of current wires instead of the continuous distribution. The final result will be obtained by a limit process. Let us consider Fig. 5.

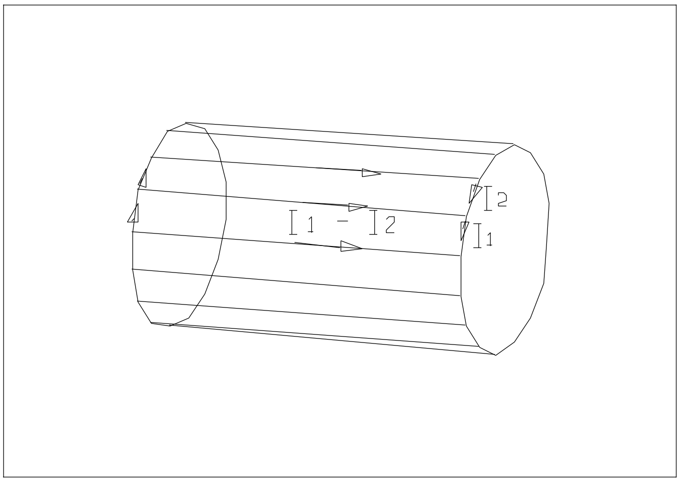

Figure 5 - Net of current wires generating multipolar potentials

We have *N* wires, leaning on the lateral surface *S*, separated by an angle

$$\Delta\theta = 2\pi/N \qquad 2.1.4)$$

and *N* wires leaning on each of the two basis. The k^{th} current on the lateral surface is placed at an angle β_k:

$$\beta_k = k\,\Delta\theta \qquad 2.1.5)$$

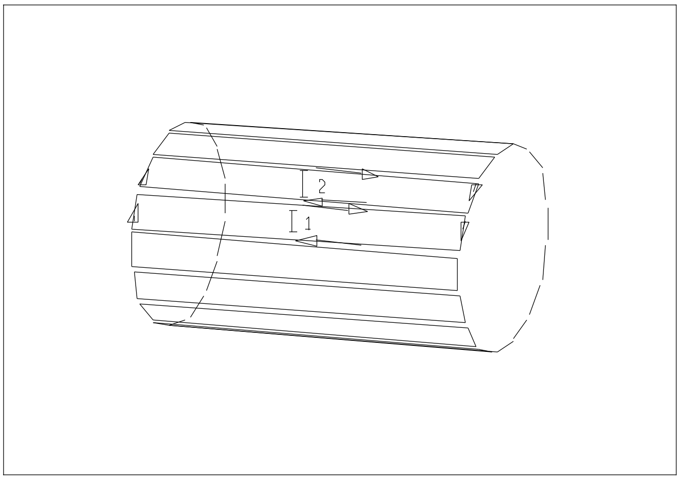

Figure 6 - Net of coils generating multipolar potentials

Figure 6 shows how the current distribution of Fig. 5 can be considered made of N rectangular coils adjacent to each other. The k^{th} coil is characterized by its average angle θ_k:

$$\theta_k = (k - \frac{1}{2}) \Delta\theta \qquad 2.1.6)$$

and its current:

$$I(\theta_k) = I_c \sin m\theta_k \qquad 2.1.7)$$

The decomposition of the current distribution in a sum of coils is useful for the potential calculation. A coil carrying a current I induces at the point $Q_o\ (r,\phi,z)$ a potential [8]:

$$\Delta P = \frac{\mu_o I}{4\pi} \Delta\omega \qquad 2.1.8)$$

where $\Delta\omega$ denotes the solid angle under which the coil is seen from $Q_o(r,\phi,z)$. Analogously to the network of currents we restrict the possible values of the Q_o coordinate ϕ to N values, namely:

$$\phi_n = n\,\Delta\theta \qquad 2.1.9)$$

The potential generated at a point $Q_o(r,\phi_n,z)$ by all the coils can be written as:

$$P(Q_o) = \frac{\mu_o I_c}{4\pi} \sum_k \Delta\omega(\theta_k,\phi_n) \sin m\theta_k \qquad 2.1.10)$$

It can be observed that:

i) for geometrical reasons the solid angle $\Delta\omega(\theta_k,\phi_n)$ depends only on the difference θ_k-ϕ_n. Through 2.1.6) and 2.1.9) we obtain:

$$\theta_k - \phi_n = (k - n - \frac{1}{2}) \Delta\theta = \theta_{k-n} \qquad 2.1.11)$$

and writing $\Delta\omega_{k-n}$ instead of $\Delta\omega(\theta_{k-n})$, the potential $P(Q_o)$ can be expressed as:

$$P(Q_o) = \frac{\mu_o I_c}{4\pi} \sum_k \Delta\omega_{k-n} \sin(m\theta_{k-n} + m\phi_n) \qquad 2.1.12)$$

ii) for geometrical reasons as well, $\Delta\omega_{k-n}$ is an even function of θ_{k-n}, therefore inside $\sin(m\theta_{k-n} + m\phi_n)$ only the even terms in θ_{k-n} contribute to the sum. The potential can be rewritten:

$$P(Q_o) = \frac{\mu_o I_c}{4\pi} \sin m\phi_n \sum_k \Delta\omega_{k-n} \cos m\theta_{k-n} \qquad 2.1.13)$$

iii) the dependence on n of the sum in 2.1.13) is equivalent to a permutation of the N terms. This is ininfluent on the sum and we can write:

$$P(Q_o) = \frac{\mu_o I_c}{4\pi} \sin m\phi_n \sum_k \Delta\omega_k \cos m\theta_k \qquad 2.1.14)$$

Inside the sum, $\Delta\omega_k$ is now the solid angle under which the coil k is seen from the point of coordinates $(r,0,z)$ and not (r,ϕ_n,z). At the limit of infinitesimal coils, $\Delta\omega_k$ becomes:

$$d\omega_k(Q_o) \dashrightarrow \frac{\partial\omega(\theta)}{\partial\theta}\, d\theta \qquad 2.1.15)$$

and the potential, writing explicitly the dependence on r, is:

$$P_m(r,\phi,z) = \frac{\mu_0 I_c}{4\pi} \sin m\phi \int_0^{2\pi} \frac{\partial\omega(r,\theta)}{\partial\theta} \cos m\theta \, d\theta \qquad 2.1.16)$$

The potential depends on ϕ only through the factor $\sin m\phi$ and this guarantees that it corresponds to a pure multipole according to *1.1.5)* and *1.1.6)*.

Looking at the deduction of the result, we observe that the key point of the cylindrical geometry is its invariance under rotation. So we can extend the result to any surface S invariant under rotation around the z axis.

The surface S is completely defined by a curve $R(z)$ $(-Z_L < z < Z_L)$. Figure 7 shows two examples with a sinusoidal $R(z)$ function. The current density $J(\theta)$ of 2.1.3) in this case depends also on z according to:

$$J(z,\theta) = \frac{mI_c}{R(z)} \cos m\theta \qquad 2.1.17)$$

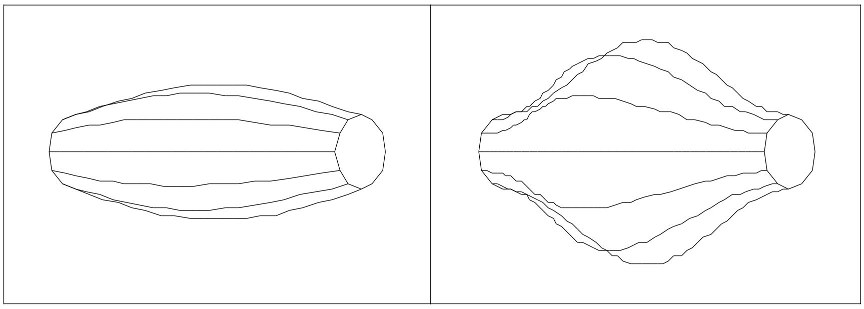

Figure 7 - Cylindrical geometries with dependence on z generating multipole potential

2.2) Current Distributions Generating Analytical Potentials

Let us show now that the function $G_{m0}(z)$ (1.1.7) can be expressed analytically if the surface S coincides with the lateral boundary of a cylinder of radius R.

According to the considerations of the previous paragraph we need to compute the solid angle $d\omega(Q_o)$ under which an infinitesimal region on the cylinder placed around a point $Q(R,\theta,Z)$ is seen from a point $Q_o(r,0,z)$. We can write:

$$d\omega(Q_0) = \frac{dS}{s^2} \cos\delta \qquad 2.2.1)$$

where dS is the infinitesimal surface around $Q(R,0,Z)$ (see Fig. 8), s the distance between Q and Q_o, and δ the angle between Q_oQ and the normal to the cylindrical surface at the point Q. The surface element on the cylinder is:

$$dS = R\, d\theta\, dZ \qquad 2.2.2)$$

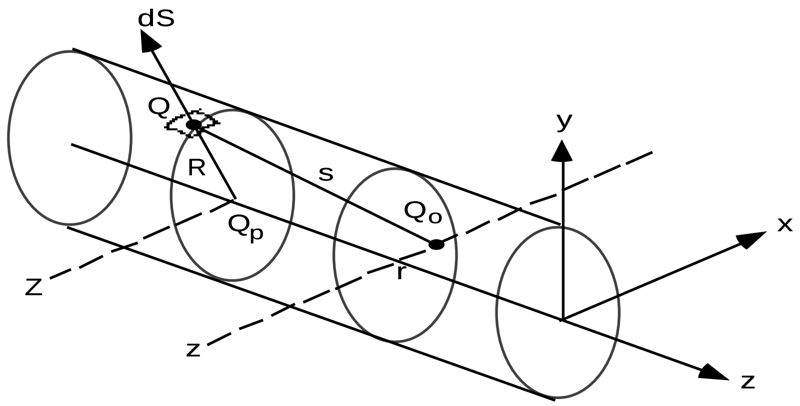

Figure 8 - Sketch of the geometry used to determine the analytical potential

Let us now introduce the point Q_p, projection of Q on the z axis.

The cartesian coordinates of Q, Q_o and Q_p are respectively $(R\cos\theta, R\sin\theta, Z)$, $(r, 0, z)$ and $(0, 0, Z)$. Naming $\boldsymbol{s}$ the vector joining Q_o to Q (s its module), $\boldsymbol{R}$ the one joining Q_p to Q (R its module) we obtain:

$$s = \sqrt{R^2 + r^2 - 2\,Rr\cos\theta + (Z-z)^2} \qquad 2.2.3)$$

$$\cos\delta = \frac{\boldsymbol{s}\cdot\boldsymbol{R}}{s\,R} = \frac{R - r\cos\theta}{s} \qquad 2.2.4)$$

and finally, substituting inside 2.2.1):

$$d\omega(Q_o) = \frac{R - r\cos\theta}{s^3}\, R\, d\theta\, dZ \qquad 2.2.5)$$

Defining:

$$g(r,\theta) = \frac{R - r\cos\theta}{s^3} \qquad 2.2.6)$$

we can write:

$$\frac{\partial\omega}{\partial\theta} = g(r,\theta)\, R\, dZ \qquad 2.2.7)$$

and eq. 2.1.16) becomes:

$$P_m(r,\phi,z) = -\frac{\mu_o I_c R}{4\pi}\sin m\phi \int_{-Z_L}^{Z_L} dZ \int_0^{2\pi} g(r,\theta)\cos m\theta\, d\theta \qquad 2.2.8)$$

Comparison between 2.2.8) and 1.1.7) gives for $G_{m0}(z)$:

$$G_{m0}(z) = \frac{\mu_o I_c R}{4\pi} \int_{-Z_L}^{Z_L} dZ \int_0^{2\pi} \cos m\theta \left(\frac{\partial^m g(r,\theta)}{\partial r^m}\right)_{r=0} d\theta \qquad 2.2.9)$$

Instead of using the definite integral on z it is convenient to use the indefinite one $G_{m0}(Z,z)$:

$$G_{m0}(Z,z) = \frac{\mu_o I_c R}{4\,\pi} \int dZ \int_0^{2\pi} \cos m\theta \left(\frac{\partial^m g(r,\theta)}{\partial r^m}\right)_{r=0} d\theta \qquad 2.2.10)$$

from which $G_{m0}(z)$ is:

$$G_{m0}(z) = G_{m0}(Z_L,z) - G_{m0}(-Z_L,z) \qquad 2.2.11)$$

The derivatives with respect to r and the integration on θ appearing in 2.2.10) are performed in the first part of Appendix A and the integration on Z in the second part. Defining:

$$t = Z - z \qquad 2.2.12)$$

$$A(t) = \sqrt{R^2 + t^2} \qquad 2.2.13)$$

$$f_h(t) = \left(\frac{t}{A}\right)^h \qquad (h = 0,1,...\infty) \qquad 2.2.14)$$

the final result is:

$$G_{m0}(t) = \mu_o I_c \frac{(2m-1)!}{R^m 4^m (m-1)!} \sum_{k=0}^{m} (-1)^k \frac{m+k+1}{2k+1} \binom{m}{k} f_{2k+1}(t) \qquad 2.2.15)$$

This formula is valid only for $m>0$.

2.3) Computation of high order terms of the field

The derivation of the high order terms of the field transverse components needs the even order derivatives of the function $G_{m0}(z)$ (2.2.15) where the $f_{2k+1}(t)$ appear. The even differentiation with respect to z is equivalent to the one with respect to t (see 2.2.12); the second order derivative with respect to t of the f_{2k+1} is performed in Appendix B. Substituting in eq. B.8) the index h with $2k+1$ we can write:

$$\frac{\partial^2 f_{2k+1}}{\partial t^2} = \frac{1}{R^2}\Big((4k^2+2k)f_{2k-1} - (12k^2+12k+3)f_{2k+1} + $$
$$+ (12k^2+18k+6)f_{2k+3} - (4k^2+8k+3)f_{2k+5}\Big) \qquad 2.3.1)$$

The 2nd order derivative of f_{2k+1} is a linear combination of functions f_{2k-1}, f_{2k+1}, f_{2k+3}, f_{2k+5}: it introduces f_{2k-1}**.** Whatever the odd initial value of $2k+1$ is, by taking successive derivatives, we end up with f_1. At this point, applying 2.3.1) we can see that no terms lower than f_1 are introduced:

$$\frac{\partial^2 f_1(t)}{\partial t^2} = \frac{-3f_1(t) + 6f_3(t) - 3f_5(t)}{R^2} \qquad 2.3.2)$$

Equation 2.3.1) suggests the definition of a matrix ***M*** associated to the second derivative. Only odd rows and columns of ***M*** must be considered and we can define:

$$M_{2k-1,2k+1} = 4k^2 + 2k$$

$$M_{2k+1,2k+1} = -(12k^2 + 12k + 3)$$

$$M_{2k+3,2k+1} = 12k^2 + 18k + 6 \qquad 2.3.3)$$

$$M_{2k+5,2k+1} = -(4k^2 + 8k + 3)$$

The sum of the elements of any column of ***M*** vanishes. Since, according to 2.2.14), asymptotically every f_{2k+1} is $+1$ at $+\infty$ and -1 at $-\infty$, all the even derivatives of f_{2k+1} vanish for $t. \to \infty$. The following table shows the first elements of ***M***.

Table I - Matrix ***M***

n	$M_{n,1}$	$M_{n,3}$	$M_{n,5}$	$M_{n,7}$	$M_{n,9}$	$M_{n,11}$	$M_{n,13}$	$M_{n,15}$	$M_{n,17}$	$M_{n,19}$	$M_{n,21}$	$M_{n,23}$
1	-3	6	0	0	0	0	0	0	0	0	0	0
3	6	-27	20	0	0	0	0	0	0	0	0	0
5	-3	36	-75	42	0	0	0	0	0	0	0	0
7	0	-15	90	-147	72	0	0	0	0	0	0	0
9	0	0	-35	168	-243	110	0	0	0	0	0	0
11	0	0	0	-63	270	-363	156	0	0	0	0	0
13	0	0	0	0	-99	396	-507	210	0	0	0	0
15	0	0	0	0	0	-143	546	-675	272	0	0	0
17	0	0	0	0	0	0	-195	720	-867	342	0	0
19	0	0	0	0	0	0	0	-255	918	-1083	420	0
21	0	0	0	0	0	0	0	0	-323	1140	-1323	506

M does not depend on the multipole order m and on the order of derivation p. In the first column $-3,6,-3$ are the coefficients of the second derivative of f_1, in the second column $6,-27,36,-15$ those of the second derivative of f_3 and so on.

Let's now introduce a vector $\boldsymbol{F}_{m0}$, associated to the function G_{m0} (see 2.2.15) with components:

$$F_{m,0,2k+1} = (-1)^k \frac{(2m-1)!}{4^m\,(m-1)!}\,\frac{m+k+1}{2k+1}\binom{m}{k} \qquad 2.3.4)$$

Functions G_{m2p} (defined in 1.1.3) are proportional to the even derivatives of G_{m0} . So, introducing the matrix $\boldsymbol{M}$, associated to the second derivative of functions f_{2k+1}, we can associate to each G_{m2p} the vectors $\boldsymbol{F}_{m2p}$:

$$\boldsymbol{F}_{m2p} = \frac{(-1)^p\, m!}{4^p(m+p)!p!}\;\boldsymbol{M}^p\;\boldsymbol{F}_{m0} \qquad 2.3.5)$$

with components:

$$F_{m,2p,2k+1} = \frac{(-1)^p\, m!}{4^p(m+p)!p!}\left(\boldsymbol{M}^p\;\boldsymbol{F}_{m0}\right)_{2k+1} \qquad 2.3.6)$$

All $F_{m,2p,2k+1}$ are zero for $k > m + 2p$, so that eq. 1.1.3) can be rewritten:

$$G_{m2p}(t) = \frac{\mu_o\, I_c}{R^{m+2p}}\sum_{k=0}^{\infty} F_{m,2p,2k+1}\, f_{2k+1}(t) \qquad 2.3.7)$$

Finally, passing from the indefinite to the finite integration from $-Z_L$ to Z_L :

$$G_{m2p}(z) = \frac{\mu_o\, I_c}{R^{m+2p}}\sum_{k=0}^{\infty} F_{m,2p,2k+1}\;[f_{2k+1}\,(Z_L - z) + f_{2k+1}(Z_L + z)] \qquad 2.3.8)$$

We can notice that while $G_{m2p}(t)$ is an odd function, $G_{m2p}(z)$ is an even function.

To obtain $B_z(r,\phi,z)$ each f_{2k+1} must be differentiated once. At Appendix B it is shown that the derivative of $f_{2k+1}(t)$, that we call $g_{2k+1}(t)$ can be written :

$$g_{2k+1}(t) = \frac{df_{2k+1}(t)}{dt} = \frac{(2k+1)R^2}{A^3}\, f_{2k}(t) \qquad 2.3.9)$$

From the above equation and definition 2.2.14) it results that $f_{2k+1}(t)$ and $g_{2k+1}(t)$ have opposite parities. Recalling eqs. 1.2.4) and 2.3.8) we can write:

$$G_{m2p+1}(z) = \frac{\mu_o\, I_c}{R^{m+2p}}\sum_{k=0}^{\infty} F_{m,2p,2k+1}\;[g_{2k+1}(Z_L + z) - g_{2k+1}(Z_L - z)] \qquad 2.3.10)$$

So $G_{m2p+1}(z)$ is odd.

From the above considerations the fundamental role of the odd functions $f_{2k+1}(t)$ and their derivatives $g_{2k+1}(t)$ appears. The behaviour of the functions $f_{2k+1}(t)$ and $g_{2k+1}(t)$ is shown in Fig. 9 for $k=1,2,3$.

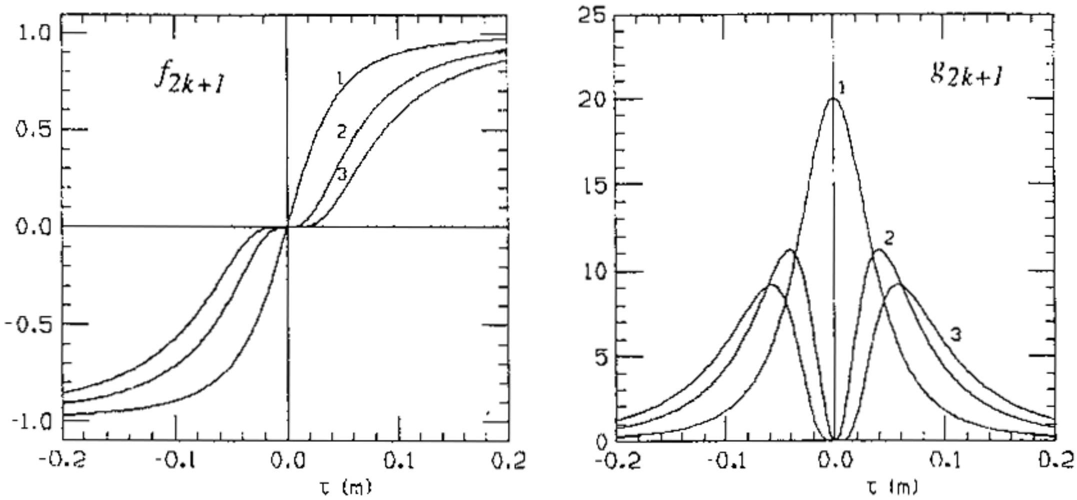

Figure 9 - Functions $f_{2k+1}(t)$ and $g_{2k+1}(t)$ for $k=1,2,3$.

Tables containing the coefficients $F_{m,2p,2k+1}$ for the dipole and the quadrupole are reported subsequently as an example which shows the spectrum of functions $f_{2k+1}(t)$ for the functions $G_{m2p}(t)$ and the spectrum of functions $g_{2k+1}(t)$ for the functions $G_{m2p+1}(t)$.

In table II the first column $(p=0)$ contains the coefficients of f_1 and f_3 in $G_{10}(z)$ except for the factor $\mu_o I_c/R$. The second column the coefficients of f_1, f_3 and f_5 in $G_{12}(z)$ except for the factor $\mu_o I_c/R^3$, etc. Passing from the dipole to the quadrupole the spectrum of f_{2k+1} inside $G_{m0}(z)$ (see first column in tables II, III) adds f_5.

Table II - Dipole coefficients $F_{1,2p,2k+1}$

k \ p	0	1	2	3
1	.5	.375	.3515625	.341796875
2	-.25	-1.21875	-2.55859375	4.31518554685
3	0	1.3125	6.796875	20.0463867185
4	0	-.46875	-8.5546875	-47.5354003905
5	0	0	5.1953125	64.0185546875
6	0	0	-1.23046875	-49.7570800785
7	0	0	0	20.8666992185
8	0	0	0	3.66577148435

Table III - Quadrupole coefficients $F_{2,2p,2k+1}$

k \ p	0	1	2	3
0	1.125	.78125	.7177734375	.692138671875
1	-1.	-3.4375	-6.5625	-10.51025390625
2	.375	5.625	22.4560546875	59.03173828125
3	0	-4.0625	-38.5546875	172.72705078125
4	0	1.09375	35.7861328125	297.568359375
5	0	0	-17.2265625	314.53857421875
6	0	0	3.3837890625	201.33544921875
7	0	0	0	-71.84912109375
8	0	0	0	10.997314453125

Figure 10 represents an example of the functions G_{2j} up to the 7^{th} order. According to 1.2.1-3, G_{20} is proportional to the basic focusing components of the fields, G_{21} to the first term of the longitudinal component, G_{22} to the pseudo-octupole components of the transverse fields, G_{23} to the second term of B_z and so on.

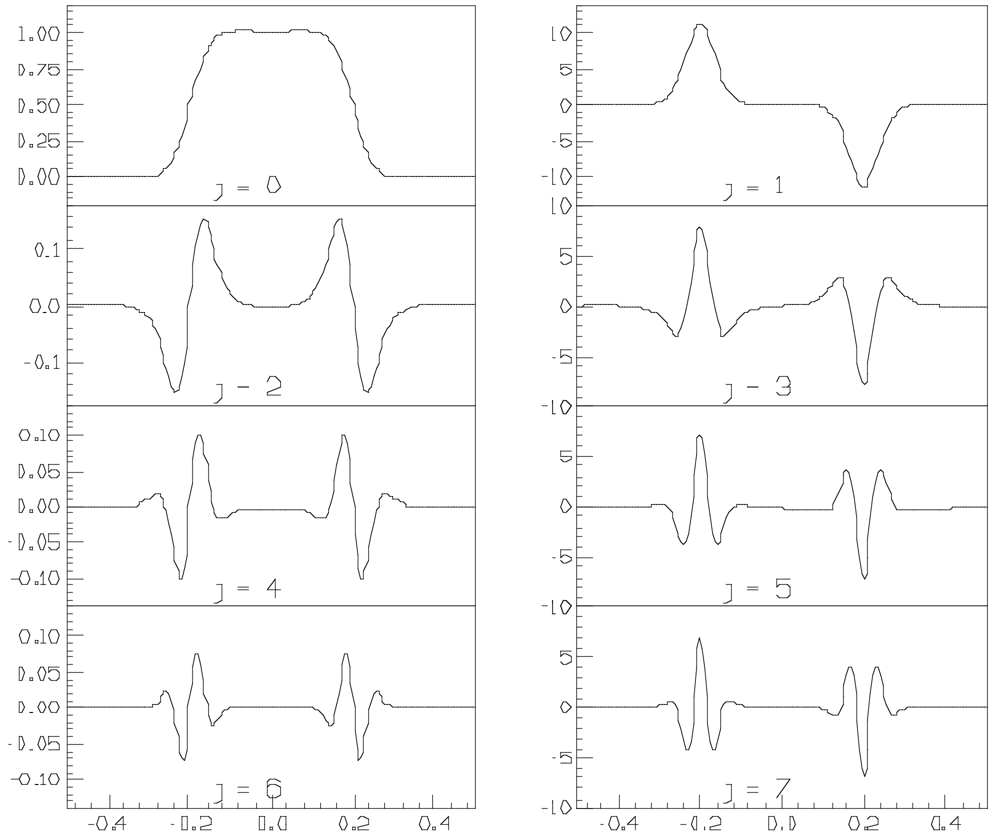

Figure 10 - Functions G_{2j} $(j=0,1,..,7)$ for a quadrupole with $Z_L = 0.2m$ and $R = 0.1m$. The factor $\frac{\mu_0 I_c}{R^{2p+2}}$ has been omitted and therefore the vertical scales are dimensionless for the even terms and (m^{-1}) for the odd ones.

3) SOLENOIDAL CASES

3.1) The *m=0* case.

For $m=0$ only the case with $\cos m\phi$ is meaningful. The potential is:

$$P_0(r,z) = \left\{G_{00}(z) + G_{02}(z)\, r^2 + G_{04}(z)\, r^4 + ...\right\} \qquad 3.1.1)$$

and the components of the magnetic field (B_ϕ is always vanishing):

$$B_r(r,z) = 2 \sum_{p=1}^{\infty} p\, G_{02p}(z)\, r^{2p-1} \qquad 3.1.2)$$

$$B_z(r,z) = \sum_{p=0}^{\infty} G_{02p+1}(z)\, r^{2p} \qquad 3.1.3)$$

The first term in B_z is proportional to $G_{01}(z)$, and in B_r proportional to $G_{02}(z)$.

Using the scheme of N coils lying on the surface S (see Fig. 5) the solenoidal case presents unusual peculiarities. Since $m=0$, the current running through adjacent coils (see Fig. 6) is always the same, the two currents of adjacent meridians compensate each other exactly and the resulting scheme becomes two end coils (TEC) with opposite currents. Nevertheless the interpretation of the current network as sum of N rectangular coils remains valid.

It could be remarked that the TEC case could be obtained simply by superposition of two coils, but in fact it is useful in our analysis for two reasons: first it is the natural extrapolation to $m=0$ of the series $m=2$ (quadrupole), $m=1$ (dipole), and second its solution is strictly linked to the solenoid case (see paragraph 3.2).

At the limit of a continuos distribution of currents adopting eq. 2.2.9) with $m=0$ and $r=0$ we obtain $G_{00}(z)$:

$$G_{00}(z) = \frac{\mu_o I_c}{4\pi} \int_{-Z_L}^{Z_L} dZ \int_0^{2\pi} \frac{R^2}{\sqrt{[R^2 + (Z-z)^2]^3}}\, d\theta \qquad 3.1.4)$$

The integration over θ gives simply a factor 2π and introducing the variable t, $G_{00}(z)$ becomes:

$$G_{00}(z) = \frac{\mu_o\ c}{2} \int_{-Z_L-z}^{Z_L-z} \frac{R^2}{\sqrt{(R^2 + t^2)^3}}\, dt \qquad 3.1.5)$$

Finally:

$$G_{00}(z) = \frac{\mu_o I_c}{2} \; [f_1(z_L - z) + f_1(z_L + z)] \qquad 3.1.6)$$

In $G_{00}(z)$ an odd f_{2k+1} appears, analogously to the cases with $m>0$. The vector $\boldsymbol{F}_{00}$ components are now deduced from expression 3.1.6):

$$F_{0,0,1} = \frac{1}{2} \quad F_{0,0,2k+1} = 0 \quad k=1,2,\infty \qquad 3.1.7)$$

Using the vector $\boldsymbol{F}_{00}$ all the formulae from 2.3.5) to 2.3.10) can be applied. We report only the most meaningful ones:

$$F_{0,2p,2k+1} = \frac{(-1)^p}{4^p (p!)^2} \left(\boldsymbol{M}^p \, \boldsymbol{F}_{00} \right)_{2k+1} \qquad 3.1.8)$$

$$G_{02p}(z) = \frac{\mu_o I_c}{R^{2p}} \sum_{k=0}^{\infty} F_{0,2p,2k+1} \; [f_{2k+1}(Z_L - z) + f_{2k+1}(Z_L + z)] \qquad 3.1.9)$$

$$G_{02p+1}(z) = \frac{\mu_o I_c}{R^{2p}} \sum_{k=0}^{\infty} F_{0,2p,2k+1} \; [-g_{2k+1}(Z_L - z) + g_{2k+1}(Z_L + z)] \qquad 3.1.10)$$

Choosing only the second term of 3.1.9) and 3.1.10) and putting $z_L=0$, i.e. placing the coil at the origin, we obtain the formulae for a single coil. We report for simplicity only $G_{C0}(z)$:

$$G_{C0}(z) = \frac{\mu_o I_c}{2} \; f_1(z) \qquad 3.1.11)$$

We observe that the absence of the two limits of 3.1.6) changes the parity of $G_{C0}(z)$ with respect to $G_{00}(z)$.

From 3.1.8) we deduce the table of coefficients $F_{0,2p,2k+1}$ valid for the Coil and the TEC cases. They are reported in table IV.

Table IV - Coefficients $F_{0,2p,2k+1}$ corresponding to the Coil and TEC

k \ p	0	1	2	3
0	.5	.375	.3515625	.341796875
1	0.	-.75	-1.875	-3.41796875
2	0	.375	3.515625	12.509765625
3	0	0	-2.8125	-22.6953125
4	0	0	.8203125	22.080078125
5	0	0	0	-11.07421875
6	0	0	0	2.255859375

3.2) Solenoid

Being I_S the total current of the solenoid, the density function $J_S(Z)$ on its lateral surface is:

$$J_S(Z) = \frac{I_S}{2Z_L} \; [u(Z+Z_L) - u(Z-Z_L)] \qquad 3.2.1)$$

We can consider a solenoid as a set of coils covering uniformly the space between $-Z_L$ and Z_L. The potential on the axis (see 3.1.11) is:

$$G_{S0}(z) = \frac{\mu_o}{2} \int_{-\infty}^{\infty} J_S(Z) \, f_1(z-Z) \, dZ \qquad 3.2.2)$$

Integrating this expression with respect to z we obtain:

$$G_{S0}(z) = \frac{\mu_o \, I_S}{2Z_L} \frac{\sqrt{R^2 + (z+Z_L)^2} - \sqrt{R^2 + (z-Z_L)^2}}{2} \qquad 3.2.3)$$

and $G_{S1}(z)$ becomes:

$$G_{S1}(z) = \frac{\mu_o I_S}{4Z_L} \; [f_1(z+Z_L) - f_1(z-Z_L)] \qquad 3.2.4)$$

By successive derivatives all $G_{S2p}(z)$ and $G_{S2p+1}(z)$ functions can be obtained, and remembering 2.3.5) and 2.3.11) we can deduce that any $G_{S2p+1}(z)$ is made up of f_{2k+1} and any $G_{S2p}(z)$, with the only exception of $G_{S0}(z)$, is made up of g_{2k+1} . To obtain the corresponding coefficients we first observe that $G_{S1}(z)$ (3.2.4) and $G_{00}(z)$ relative to the TEC (3.1.6) have the same behaviour, and we can write for $p = 0,1,2,...$:

$$G_{S2p+1}(z) = \frac{dG_{S2p}(z)}{dz} = \frac{\mu_o \, I_s}{2R^{2p} \, Z_L} \sum_{k=0}^{\infty} F_{0,2p,2k+1} \, [f_{2k+1}(z+Z_L) - f_{2k+1}(z-Z_L)] \qquad 3.2.5)$$

and by derivation with respect to z:

$$\frac{d^2 G_{S2p}(z)}{dz^2} = \frac{\mu_o \, I_s}{2R^{2p} \, Z_L} \sum_{k=0}^{\infty} F_{0,2p,2k+1} \, [g_{2k+1}(z+Z_L) - g_{2k+1}(z-Z_L)] \qquad 3.2.6)$$

Using eq. 1.1.3) we deduce:

$$G_{S2p+2}(z) = (-1)^{p+1} \frac{1}{4^{p+1}(p+1)!^2} \frac{d^{2p+2} G_{S0}}{dz^{2p+2}} = - \frac{1}{4(p+1)^2} \frac{d^2 G_{S2p}(z)}{dz^2} \qquad 3.2.7)$$

and eq. 3.2.6) can be rewritten:

$$G_{S2p+2}(z) = - \frac{1}{4(p+1)^2} \frac{\mu_o \, I_s}{2R^{2p} Z_L} \sum_{k=0}^{\infty} F_{0,2p,2k+1} \, [g_{2k+1}(z+Z_L) - g_{2k+1}(z-Z_L)] \qquad 3.2.8)$$

and introducing for simplicity new coefficients $D_{0,2p+2,2k+1}$ defined as:

$$D_{0,2p+2,2k+1} = - \frac{1}{4(p+1)^2} F_{0,2p,2k+1} \qquad 3.2.9)$$

$$G_{S2p+2}(z) = \frac{\mu_o I_s}{2R^{2p} Z_L} \sum_{k=0}^{\infty} D_{0,2p+2,2k+1} [g_{2k+1}(z+Z_L) - g_{2k+1}(z-Z_L)] \qquad 3.2.10)$$

from which:

$$G_{S2p}(z) = \frac{\mu_o I_s}{2R^{2p-2} Z_L} \sum_{k=0}^{\infty} D_{0,2p,2k+1} [g_{2k+1}(z+Z_L) - g_{2k+1}(z-Z_L)] \qquad (p>0) \quad 3.2.11)$$

Let us summarize: the functions G_{S2p+1} are obtained with the coefficients $F_{0,2p,2k+1}$ reported in table IV, while the functions G_{S2p} contain the modified coefficients $D_{0,2p,2k+1}$. The parity of the solenoidal function is opposite to the one of the multipoles with $m > 0$.

Figure 11 represents an example of the functions $G_{Sj}(z)$ up to the 7th order for a solenoid with $Z_L = 0.2\ m$ and $R = 0.05\ m$. The functions with odd j determine the longitudinal component of the field, those with even j determine the radial component. From the point of view of field components G_{S0} can be ignored.

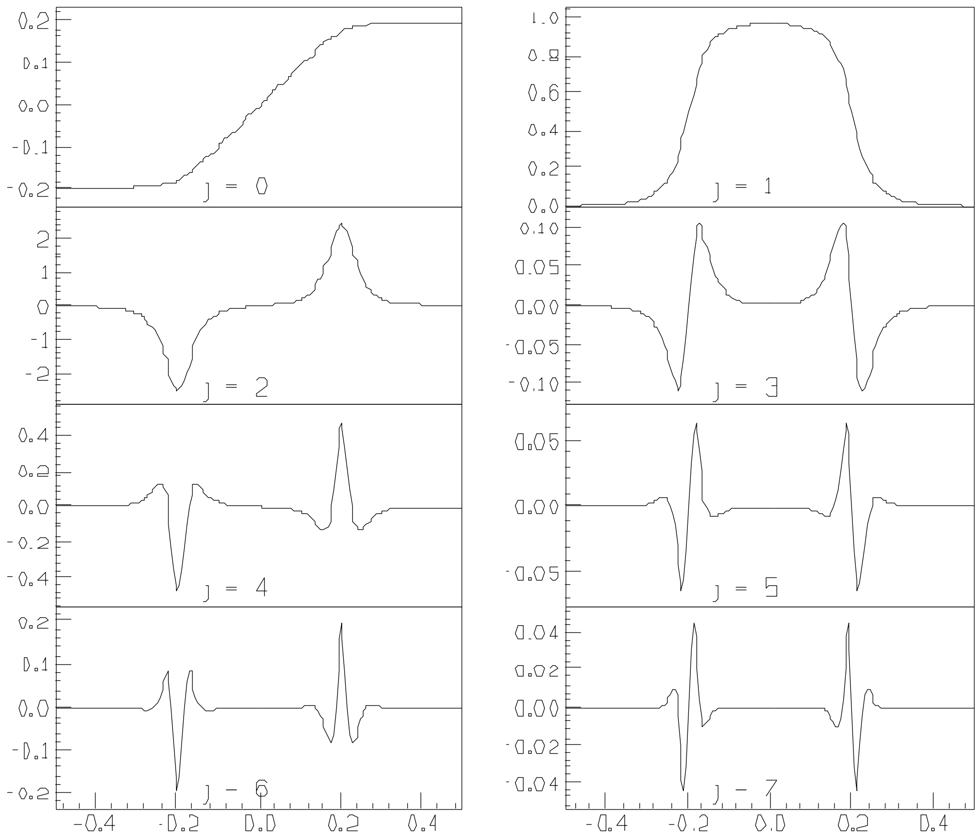

Figure 11 - Functions G_{Sj} $(j=0,1,..7)$ for a solenoid with $Z_L = 0.2\ m$ and $R = 0.05\ m$. They are plotted normalized to the factor $\frac{\mu_o I_s}{2R^n Z_L}$ and therefore the vertical scales are dimensionless for odd j and (m^{-1}) for even j.

3.3) Comparison With Other Solutions for the Coil

While our analytical solutions for dipoles, quadrupoles and in general for $m>0$ are new (as far as we know), for the coil case analytical formulae are already known in the literature. For instance in [9] two expression of the vector potential originated by a coil are given in polar coordinates using elliptical integrals or spherical harmonics. A third solution in cylindrical coordinates expressed as function of Bessel functions is explicated in [8]:

$$A_\phi = \frac{R\,\mu_O I_S}{\pi} \int_0^\infty K_1(kR)\, I_1(kr) \cos kz \, dk \qquad r<R \qquad 3.3.1)$$

$$A_\phi = \frac{R\,\mu_O I_S}{\pi} \int_0^\infty I_1(kR)\, K_1(kr) \cos kz \, dk \qquad r>R \qquad 3.3.2)$$

To show the equivalence between our solution for $r<R$ and the one expressed by 3.3.1) we will obtain the scalar potential from the vector one. We first deduce B_z from 3.3.1):

$$B_z = \frac{\delta A_\phi}{\delta r} + \frac{A_\phi}{r} = \frac{R\mu_O I_S}{\pi} \int_0^\infty K_1(kR) \left(\frac{I_1}{r} + \frac{dI_1}{dr} \right) \cos kz \, dk \qquad 3.3.4)$$

and successively by integration on z we obtain a potential that we call $P_{SB}(r,z)$ (B for Bessel).

$$P_{SB}(r,z) = \frac{R\mu_O I_S}{\pi} \int_0^\infty K_1(kR) \left(\frac{I_1}{r} + \frac{dI_1}{dr} \right) \frac{\sin kz}{k} \, dk \qquad 3.3.5)$$

We recall that the potential near the region $r=0$ is defined by the potential on the axis. From well known properties of the function $I_1(kr)$ (see for instance [10] p. 375) we deduce:

$$\left(\frac{I_1}{r} + \frac{dI_1}{dr} \right)_{r=o} = k \qquad 3.3.6)$$

and from 3.3.5)

$$P_{SB}(0,z) = \frac{R\mu_O I_S}{\pi} \int_0^\infty K_1(kR) \sin kz \, dk \qquad 3.3.7)$$

$P_{SB}(0,z)$ coincides with $G_{S0}(z)$ and consequently $P_{SB}(r,z)$ coincides with $G_S(r,z)$ if:

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{2R}{\pi} \int_0^\infty K_1(kR) \sin kz \, dk \qquad 3.3.8)$$

and this is shown in Appendix C.

4. FINAL CONSIDERATIONS

The formalism which describes the magnetic potential near the axis of a multipole, developed in [1,2,3,4] and recalled in the first paragraph, has been largely used in this paper. This approach, in spite of the simplicity of the proof, is enormously powerful.

We have demonstrated that cylindrical distributions of currents can be managed analytically and generate pure 3D magnetic multipoles. The general formulae for any order have been obtained; the specific formulae relative to dipoles and quadrupoles have been developed in detail. The dipole formulae correspond to rectangular dipoles. Nevertheless we observe that being them valid for any length we can obtain the magnetic field of a sector dipole as a sum of very thin dipole-quadrupole distributed along the arc. The only inconvenient of this solution, which satisfies Maxwell equations, is the long computing time.

The solenoidal cases, coil and solenoid have been treated, with some caution, as limit case of our formulae for $m=0$. Since the coil case has been already treated analytically in the literature using spherical and cylindrical functions, the equivalence of our solution with the one which uses modified Bessel functions has been shown in a separated paragraph.

The cylindrical model here developed are very useful to represent the magnetic elements of accelerators in simulation codes to study the effect of non-linearities on beam dynamics. Once known the field behaviour of an element, either by 3-D codes, either by magnetic measurement, it is possible to approximate the data one the function G_{m0} or a combination of different G_{m0}, just playing with the parameters R, Z_L to fit the shape, and I_c to fit the value. From there on the field is analytically determined up to any order, and this provides a wide flexibility in the codes.

A first application has been the study of the interaction regions of DAΦNE, the Frascati Φ–factory storage ring [11,12]. Due to the beam-beam crossing angle, the beams pass off-axis in low-beta quadrupoles and solenoids. Representing the fields with our analytical formulae it has been possible to study the modification of the linear optics and of the on-axis compensation scheme. Also the effect of the pseudo-octupole present in the quadrupole fringing region [7,13] is included.

Some examples of how the magnetic elements have been represented are shown in the following figures: Figure 12 refers to the permanent magnet quadrupole that will be installed in DAΦNE low-beta region; the behaviour of the measured linear gradient is compared to the function G_{20} fitting it. The slight asymmetry of the data is due to a geometrical asymmetry of the quadrupole necessary to fit the detector space requirements. The two parameters which have been used to fit the data are $R = 0.08\ m$ and $Z_L = 0.10\ m$, $I_c = 30\ kA$.

Figure 13 represents the same curves referred to one of the normal conducting quadrupole of the ring. In this case the asymmetry of the data is due to measurement errors. $R = 0.10\ m$, $Z_L = 0.15\ m$ and $I_c = 66\ kA$ have been used.

In Fig. 14 the longitudinal component of the magnetic field in one of the compensator solenoids is represented; the stars correspond to the field calculated with a 3-D magnetic code, the solid lines correspond to the field obtained with a combination of two cylindrical solenoids centered in the same position, which fit reasonably the bumps in the field behaviour. In fact the two function parameters are: $R_1 = 0.128\ m$, $R_2 = 0.265\ m$, $Z_{L1} = 0.41\ m$, $Z_{L2} = 0.28\ m$; $I_2 = -\ 0.26\ I_1$, $I_1 = 156\ kA$.

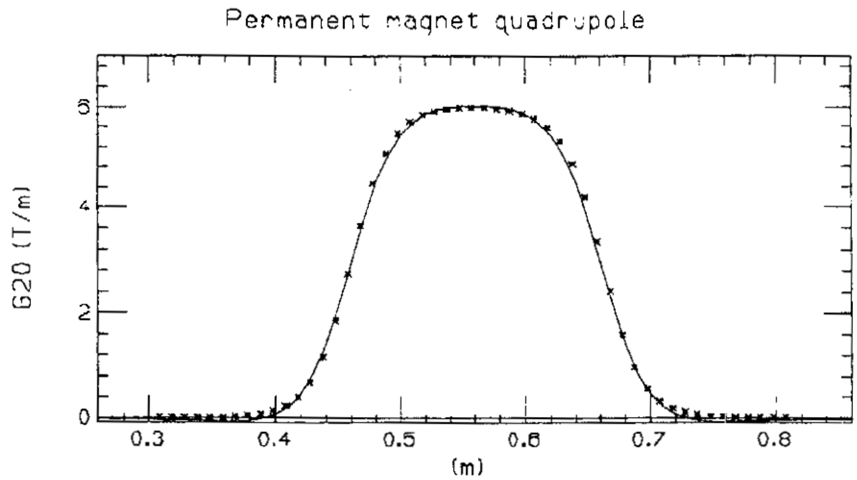

Figure 12 - DAΦNE permanent magnet quadrupole - ****** measured linear gradient;
——— function G_{20}.

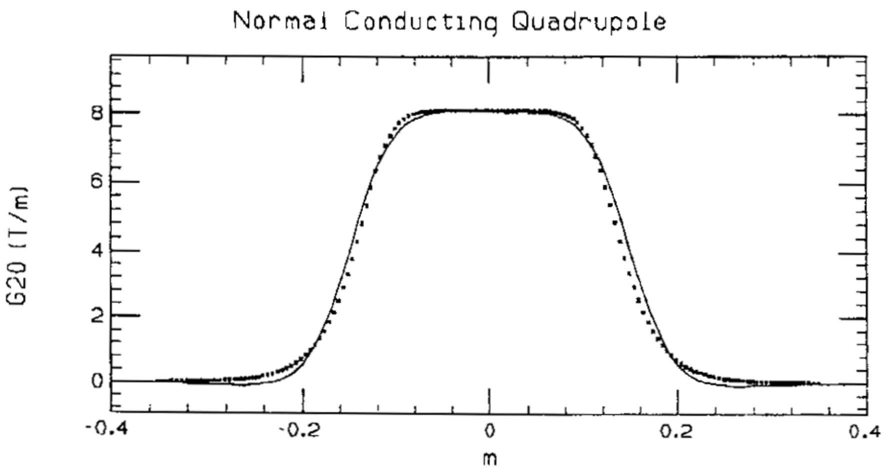

Figure 13 - DAΦNE iron quadrupole - ****** measured linear gradient;
——— function G_{20}.

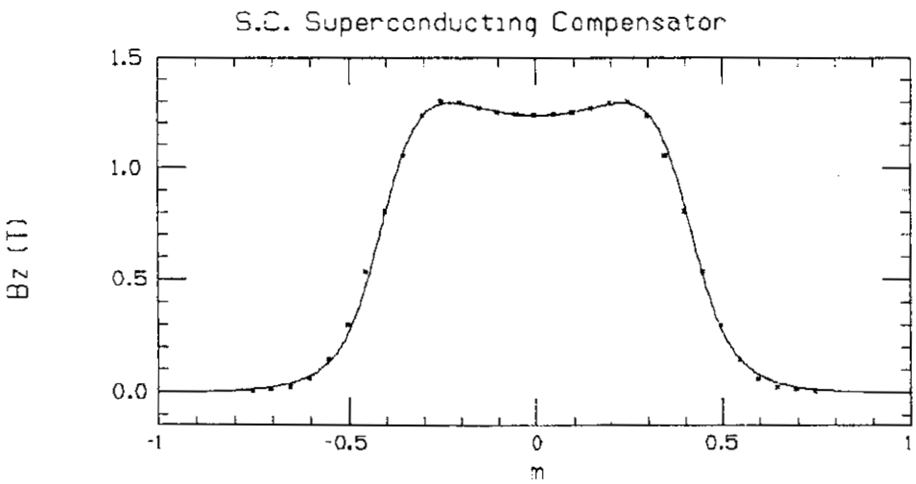

Figure 14 - DAΦNE compensating solenoid - ****** 3D code computed B_z;
——— B_z obtained with the combination of two G_{S0}

ACKNOWLEDGMENTS

The interest of the first author about fringing fields was lit up by enlightnening discussions with C. Iselin of CERN during the summer of 1993.

This work would not have been possible without an initial work done by means of a few computer codes treating the field generated by fixed distributions of currents. It is a pleasure for the authors to thank C. Milardi for the collaboration in this phase of the work.

The authors are grateful to M. Preger and M. Serio of LNF, B. Zotter and W. Scandale of CERN and G. Fano of Bologna university, for a careful reading of the paper.

Special thanks go to P. Possanza who has skillfully and patiently edited the manuscript.

REFERENCES

[1] S. Caspi, M. Helm, and L.J. Laslett, "3D Field Harmonics" Lawrence Berkeley Laboratory-30313. March 1991.

[2] S. Caspi, M. Helm, and L.J. Laslett and V.O. Brady, "An approach to 3D Magnetic Field Calculations Using Numerical and Differential Algebra Methods" Lawrence Berkeley Laboratory-32624. July 1992.

[3] S. Caspi, M. Helm, and L.J. Laslett, "The Use of Harmonics in 3-D Magnetic Fields Harmonics". IEEE Transaction on MAGNETICS Vol. 30, Num. 4 p. 2419, July 1994.

[4] G. Leleux, "Complements sur la Physique des Accelerateurs" Laboratoire National Saclay /86-101 March 1986.

[5] E. Regenstreif, Proceeding of the Los Alamos Linac Conference (1966).

[6] R.L. Gluckstern, Brookhaven National Laboratory AADD 122 (1966).

[7] J. Le Duff, "Excitation of Non Linear Resonances by the Fringing Field of the Insertion Quadrupoles in DORIS". DESY H 72/2 June 1974.

[8] Wolfgang K.H. Panofsky, Melba-Phillips, "Classical Electricity and Magnetism". Addison-Wesley Series in Physics.

[9] John D. Jackson, "Classical Electrodynamics" J.Whiley edit. p. 178.

[10] Handbook of mathematical functions. Edited by Milton Abramowitz and Irene A. Stegun, p. 376, 9.6.25.

[11] C. Biscari, "Low Beta Quadrupole Fringing Field on Off-Axis Trajectory", Proceedings of Arcidosso. Workshop Sept. 1994 - to be published.

[12] M. Bassetti, C. Biscari, "Off-axis fields in DAΦNE Interaction Regions", in preparation.

[13] P. Krejcik, "Nonlinear quadrupole end-field effects in CERN antiproton accumulators", PAC 87 Vol. 2, p. 1278-1280.

[14] Unbestimmte Integrale - Erster Teil. Springer-Verlag.

APPENDIX A.

A1) m^{th} derivative of g(r) and integration over θ of $G_{m0}(Z,z)$

Let us consider the function:

$$g(r) = \frac{R - r\cos\theta}{\sqrt{[R^2 + r^2 - 2Rr\cos\theta + (Z-z)^2]^3}} \qquad A1.1)$$

Defining:

$$s = \sqrt{R^2 + r^2 - 2Rr\cos\theta + (Z-z)^2} \qquad A1.2)$$

$$f(r) = R - r\cos\theta \qquad A1.3)$$

$$h(r) = r - R\cos\theta \qquad A1.4)$$

we can simply write:

$$g(r) = \frac{f(r)}{s^3} \qquad A1.5)$$

Denoting by a prime the derivative with respect to *r*, we get:

$$f' = -\cos\theta \qquad A1.6)$$

$$h' = 1 \qquad A1.7)$$

$$s' = \frac{h}{s} \qquad A1.8)$$

From the above equations, putting $r = 0$ we obtain:

$$A(Z-z) = s(0) = \sqrt{R^2 + (Z-z)^2} \qquad A1.9)$$

$$f(0) = R \qquad A1.10)$$

$$g(0) = \frac{R}{A^3} \qquad A1.11)$$

$$h(0) = -R\cos\theta \qquad A1.12)$$

By differentiating *g(r) m* times, we get terms whose dependence on θ is proportional to $\cos^n\theta$ with *n* between *0* and *m*. If we look at the integration over θ of 2.2.9) we already have a term $\cos m\theta$. The integrals to be performed have the form:

$$T(m,n) = \int_0^{2\pi} \cos m\theta \cos^n\theta \, d\theta \qquad A1.13)$$

Considering that:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad A1.14)$$

and taking the n^{th} power of A1.14) the sum of the first and the last terms gives $\frac{\cos n\theta}{2^{n-1}}$. The other terms contain a lower integer multiple of θ. So we obtain:

$$T(m,n) = \frac{1}{2^{m-1}}\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta + const \int_0^{2\pi} \cos m\theta \cos(n-2)\theta \, d\theta + \cdots \qquad A1.15)$$

T(m,n) does not vanish only if n is equal to its maximum value *m*:

$$T(m,m) = \frac{1}{2^{m-1}}\int_0^{2\pi} \cos^2 m\theta \, d\theta = \frac{\pi}{2^{m-1}} \qquad A1.16)$$

We can now face the problem of determining $\frac{\partial^m g(r)}{\partial r^m}$. As shown in A1.11), *g(0)* does not contain any factor *cos θ*. A factor cos *θ* is generated only if one takes the derivative of either the function *s* or the function *f*. Since *f ''* vanishes the derivative must be like:

$$\frac{\partial^m g}{\partial r^m} = \frac{a_m f\, h^m}{s^{2m+3}} + \frac{b_m f'\, h^{m-1}}{s^{2m+1}} \qquad A1.17)$$

The coefficient a_m is the product with alternating sign of the odd numbers starting from *1*, namely:

$$a_m = (-1)^m \prod_{p=1}^{p=m} (2p+1) \qquad A1.18$$

In the mathematical literature for simplicity the two following definitions are routinely used:

$$(2m+1)!! = \prod_{p=1}^{p=m}(2p+1) \qquad A1.19$$

$$(2m)!! = \prod_{p=1}^{p=m}(2p) \qquad A1.20)$$

As the function *p!!* is not universally known, in the text we will use the equivalent formula:

$$(2m)!! = 2^m\, m! \qquad A1.21)$$

$$(2m+1)!! = \frac{(2m+1)!}{(2m)!!} = \frac{(2m+1)!}{2^m\, m!} \qquad A1.22)$$

b_m can be derived by keeping separated the terms that are deduced from the derivation of *f*. We obtain therefore:

$$\frac{\partial g}{\partial r} = - \frac{3\, f\, h}{s^5} + \frac{f\,'}{s^3}$$

$$\frac{\partial^2 g}{\partial r^2} = \left(\frac{15\, f\, h^2}{s^7} - \frac{3\, f\,'h}{s^5} \right) - \frac{3\, f\,'h}{s^5} \qquad A1.23)$$

$$\frac{\partial^3 g}{\partial r^3} = \left(- \frac{105\, f\, h^3}{s^9} + \frac{15\, f\,'\, h^2}{s^7} \right) + \frac{15\, f\,'\, h^2}{s^7} + \frac{15\, f\,'\, h^2}{s^7}$$

Every differentiation adds a new term equal to the ones already obtained. Clearly after *m* derivations we find:

$$b_m = m\, a_{m-1} = m\, (-1)^{m-1}\, (2m-1)!! \qquad A1.24)$$

and grouping the two terms:

$$\frac{\partial^m g}{\partial r^m} = (-1)^m\, h^{m-1} \left(\frac{(2m+1)!!f\, h}{s^{2m+3}} - \frac{m(2m-1)!!f\,'}{s^{2m+1}} \right) \qquad A1.25)$$

Substituting to the functions their value for $r = 0$, and using the definition A1.9) for $s(0)$ one obtains:

$$\left(\frac{\partial^m g}{\partial r^m} \right)_{r=0} = \cos^m\theta \left(\frac{(2m+1)!!\, R^{m+1}}{A^{2m+3}} - \frac{m(2m-1)!!R^{m-1}}{A^{2m+1}} \right) \qquad A1.26)$$

If we multiply A1.22) by $\cos m\theta$ and take into account eq. A1.16) and A1.22) we obtain for the integral over θ of eq. 2.2.9):

$$\int_0^{2\pi} \cos m\theta \left(\frac{\partial^m g}{\partial r^m} \right)_{r=0} d\theta = \frac{4\pi(2m-1)!R^{m-1}}{(m-1)!4^m} \left(\frac{(2m+1)R^2}{A^{2m+3}} - \frac{m}{A^{2m+1}} \right) \qquad A1.27)$$

A2) Integration on Z of $G_{m0}(Z,z)$

Let us now face the integration over *Z* of eq. 2.2.10). By using A1.27) it can be rewritten:

$$G_{m0}(Z,z) = \frac{\mu_0\, I_c\, (2m-1)!R^{m-1}}{4^m(m-1)!} \int \left(\frac{(2m+1)R^2}{A^{2m+3}} - \frac{m}{A^{2m+1}} \right) dZ \qquad A2.1)$$

Recalling:

$$t = Z - z \qquad A2.2)$$

$$A(t) = \sqrt{R^2 + t^2} \qquad A2.3)$$

let us consider the integral:

$$I_m(t) = \int \left(\frac{(2m+1)R^2}{A^{2m+3}} - \frac{m}{A^{2m+1}} \right) dt \qquad A2.4)$$

The two integrals of A2.4) have the following form:

$$F_m = \int \frac{dt}{A^{2m+1}} \qquad A2.5)$$

Defining :

$$f_m(t) = \left(\frac{t}{A}\right)^m \qquad A2.6)$$

the integral A2.5) can be written as [14]:

$$F_m = \frac{1}{R^{2m}} \sum_{k=0}^{m-1} \frac{(-1)^k f_{2k+1}(t)}{2k+1} \binom{m-1}{k} \qquad A2.7)$$

Applying eq. A2.7) the total integral in A2.1) becomes:

$$I_m(t) = \frac{1}{R^{2m}} \left(\sum_{k=0}^{m-1} \left(\frac{2m+1}{2k+1} \binom{m}{k} - \frac{m}{2k+1} \binom{m-1}{k} \right) (-1)^k f_{2k+1} + (-1)^m f_{2m+1} \right) \qquad A2.8)$$

Each coefficient inside the sum, using well known properties of binomial coefficients, can be written as:

$$\frac{(2m+1)}{(2k+1)} \binom{m}{k} - \frac{m}{2k+1} \binom{m-1}{k} = \frac{(m+k+1)}{(2k+1)} \binom{m}{k} \qquad A2.9)$$

and therefore:

$$I_m(t) = \frac{1}{R^{2m}} \left(\sum_{k=0}^{m-1} (m+k+1) \binom{m}{k} (-1)^k \frac{f_{2k+1}}{2k+1} + (-1)^m f_{2m+1} \right) \qquad A2.10)$$

If $k = m$ the result of A2.9) is equal to 1. Therefore we can include the last term $(-1)^m f_m(t)$ of A2.8) inside the sum provided we extend it up to m and then:

$$I_m(t) = \frac{1}{R^{2m}} \sum_{k=0}^{m} (-1)^k \frac{m+k+1}{2k+1} \binom{m}{k} f_{2k+1}(t) \qquad A2.11)$$

and:

$$G_{m0}(t) = \frac{\mu_0 I_c (2m-1)!}{4^m (m-1)!\, R^m} \sum_{k=0}^{m} (-1)^k \frac{m+k+1}{2k+1} \binom{m}{k} f_{2k+1}(t) \qquad A2.12)$$

APPENDIX B.

Calculation of first and second derivative of $f_h(t)$

Recalling the definitions A2.3) and A2.6):

$$A(t) = \sqrt{R^2 + t^2} \qquad B.1)$$

$$f_h = \left(\frac{t}{A}\right)^h \qquad B.2)$$

and by using the partial result:

$$\frac{\partial A}{\partial t} = \frac{t}{A} = f_1 \qquad B.3)$$

it is easy to deduce:

$$\frac{\partial f_h}{\partial t} = h \frac{R^2}{A^3} f_{h-1} \qquad B.4)$$

and differentiating again:

$$\frac{\partial^2 f_h}{\partial t^2} = \frac{1}{R^2}\left(\frac{h(h-1)R^6}{A^6} f_{h-2} - \frac{3hR^4}{A^4} f_h \right) \qquad B.5)$$

In order to eliminate the powers A^6 and A^4 B.1) suggests to replace R^2 with $(A^2 - t^2)$ obtaining:

$$R^4 = A^4 - 2A^2 t^2 + t^4 \qquad B.6)$$

$$R^6 = A^6 - 3A^4 t^2 + 3A^2 t^4 - t^6 \qquad B.7)$$

and then:

$$\frac{\partial^2 f_h}{\partial t^2} = \frac{(h^2-h)f_{h-2} - 3h^2 f_h + (3h^2+3h)f_{h+2} - (h^2+2h)f_{h+4}}{R^2} \qquad B.8)$$

APPENDIX C.

Equivalence of the two coil solutions

We must prove the validity of eq. 3.3.8):

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{2R}{\pi}\int_0^\infty K_1(kR)\sin kz\, dk \qquad C.1)$$

Let us introduce an integration on *z*:

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{2R}{\pi}\int_0^z ds \int_0^\infty K_1(kR)\, k\cos ks\, dk \qquad C.2)$$

and the integral representation of $K_1(kR)$ ([10]:

$$K_1(kR) = R\int_0^\infty \frac{\cos kt}{k\sqrt{(R^2+t^2)^3}}\, dt \qquad C.3)$$

C.2) becomes:

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{2R^2}{\pi}\int_0^z ds \int_0^\infty \frac{dt}{\sqrt{(R^2+t^2)^3}} \int_0^\infty \cos kt\cos ks\, dk \qquad C.4)$$

The integral on *k* can be written as:

$$\int_0^\infty \cos kt\cos ks\, dk = \frac{1}{2}\int_0^\infty [\cos k(s+t) + \cos k(s-t)]\, dk = \frac{\pi}{2}\,[\delta(s+t) + \delta(s-t)] \qquad C.5)$$

Only $\delta(s-t)$ is effective and C.4) becomes:

$$\frac{z}{\sqrt{R^2+z^2}} = R^2\int_0^z \frac{ds}{\sqrt{(R^2+t^2)^3}} \qquad C.6$$

which is easily proven by derivation.